Lecture 3 — Mon Jan/29/2024

Last time
- Characteristic Functions
- Levy's inversion formula

Today
- Weak convergence
- Levy's Convergence Theorem

Weak convergence

We use $\mathcal{P}(\mathbb{R})$ to denote probability measures on $\mathbb{R}$, and $C_b(\mathbb{R})$ to denote bounded continuous functions on $\mathbb{R}$.

Def: Consider a sequence $(\mu_n)_n \subseteq \mathcal{P}(\mathbb{R})$ and $\mu \in \mathcal{P}(\mathbb{R})$. We say that μ_n **converges weakly** to μ iff

$$\int f(x)\,d\mu_n(x) \longrightarrow \int f(x)\,d\mu(x) \quad \forall f \in C_b(\mathbb{R}).$$

This is often written as $\mu_n \xrightarrow{w} \mu$. $\dashv$

Remark. We identify μ with its probability distribution $F(x) := \mu(-\infty, x]$, and a random

Skorokhod Representation Theorem

variable $X \sim \mu$. Thus, we also write

$$F_n \xrightarrow{w} F \quad \text{and} \quad X_n \xrightarrow{w} X.$$

A more practical characterization is:

Lemma: Let (F_n) a secuence of distribution functions then $F_n \xrightarrow{w} F$ iff

$$\lim_n F_n(x) = F(X)$$

for all point of continuity x of F. ⊣

Exercise: Proof this Lemma (it follows by a limsupery argument).

Remark: Continuity is important. Consider $X_n = \frac{1}{n}$ and $X = 0$. Then

$$\lim \mu_{X_n}(f) = \lim f(\tfrac{1}{n}) = f(0) = \mu_X(f).$$

BUT $F_{X_n}(0) \to 0$ while $F_X(0) = 1$.

Theorem (Helly's selection principle)

For every seguence of distributions $(F_n)_n$ there is a subseguence $(F_{n_i})_i$ and a right semicontinuous function F such that

$$\lim_{i \to \infty} F_{n_i}(x) = F(x)$$

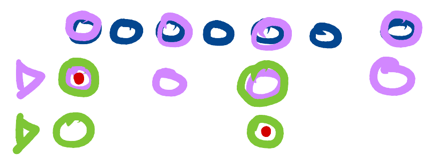
for all point of continuity x of F. $\dashv$

Proof: We construct F via a diagonalization. Let $\{q_1, q_2, \ldots\} = \mathbb{Q} \subseteq \mathbb{R}$ be a countable dense set. For each k, $F_m(q_k) \in [0,1]$ $\forall m$. Thus, there exists a subsequence $m_1(i)$ such that

$$F_{m_1(i)}(q_1) \to H(q_1).$$ ← Some value

We can then take a subsequence of $(m_1(i))_i$ named $(m_2(i))_i$ such that

$$F_{m_2(i)}(q_2) \to H(q_2).$$



We recursively define $(m_k(i))_i$ $\forall k$. Consider the sequence $F_{m(k)} = F_{m_k(k)}$ converges for any $q \in \mathbb{Q}$, i.e. $F_{m(k)} \to H(q)$.
Extend H to a function via

$$F(x) := \inf\{H(q) : q \in \mathbb{Q},\ q > x\}.$$

This is a right continuous function (Why?).

To finish, let x be a continuity point of F. Pick $a, b, c \in \mathbb{Q}$ with $a < b < c$ and

$$F(x) - \varepsilon < F(a) \le F(b) \le F(x) \le F(c) < F(x) + \varepsilon$$

Since $F_{n(k)}(b) \to F(b) \geq F(a)$ and $F_{n_{(k)}}(c) \to F(c)$

we have that for large k

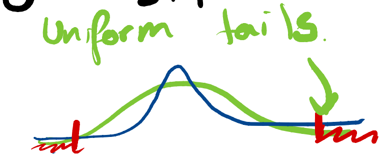
$$F(x) - \varepsilon < F_{n(k)}(b) \leq F_{n(k)}(x) \leq F_{n(k)}(c) < F(x) + \varepsilon.$$

Since ε is arbitrary, the result follows. $\square$

But is it true that F is a distribution?

Not always.

Def: A sequence of dist. f. (F_n) is **tight** if for all ε, there exists a $K_\varepsilon > 0$ s.t. s.t.



$$1 + F_n(-K_\varepsilon) - F_n(K_\varepsilon) \leq \varepsilon \quad \forall n.$$

Theorem: A subsequential limit is a distribution function if, and only if, the sequence (F_n) is tight.

Proof: Suppose $F_{n(k)} \xrightarrow{w} F$ and (F_n) is tight. Let $a < -K_\varepsilon$ and $b > K_\varepsilon$ be continuity points of F. Then

$$1 - F(a) - F(b) = \lim 1 - F_{n(k)}(a) + F_{n(k)}(b)$$
$$\leq \varepsilon.$$

This implies that $\limsup_{x \to \infty} 1 - F(x) + F(-x) \leq \varepsilon$, which implies that F is a dist. f. ← Why?

To prove the converse, suppose F_n is not tight. Thus, $\exists \varepsilon$ and a subsequence $n(k) \to \infty$ such that

$$1 - F_{n(k)}(k) + F_{n(k)}(k) \geq \varepsilon \quad \forall k.$$

WLOG $F_{n(k)} \xrightarrow{w} F$. Let $a < 0 < b$ be continuity points of F, then

$$\begin{aligned} 1-F(b) + F(a) &= \lim_{k\to\infty} 1-F_{n(k)}(b) + F_{n(k)}(a) \\ &= \liminf_{k} 1 - F_{n(k)}(k) + F_{n(k)}(-k) \\ &\geq \varepsilon. \end{aligned}$$

Taking $b \to \infty$ and $a \to -\infty$, we have that $\lim_{x\to\infty} F(x) \neq 1$ or $\lim_{x\to-\infty} F(x) \neq 0$. $\square$

Lévy's Convergence Theorem

Theorem: Let (F_n) be a sequence of dens. f, and let φ_n be the ch. f. of F_n. Suppose that

$$g(\theta) := \lim_{n\to\infty} \varphi_n(\theta) \text{ exists } \forall \theta \in \mathbb{R},$$

and that g is cont. at 0. Then $g = \varphi_F$ for some distribution F and

$$F_n \xrightarrow{w} F.$$

Proof We start by noticing that the converse is true: If $F_n \xrightarrow{w} F$, then by def. $\varphi_{F_n}(\theta) \to \varphi_F(\theta)$.

integral of bounded cont function.

Now, let's assume for a moment that F_n is tight. Then, Helly's selection Theorem tell us $\exists (F_{n(k)})$ and a dist. f. such that

$$F_{n(k)} \xrightarrow{w} F.$$

Then, we would have $\varphi_{n_k}(\theta) \to \varphi_F(\theta)$ $\forall n$.

Thus, $g = \varphi_F$.

Searching contradiction, assume F_n does not converge to F. Thus $\exists\, x$ a cont point of F and a subsequence $(F_{m(k)})$ s.t.

$$|F_{m(k)}(x) - F(x)| \geq \eta \quad \forall k. \quad (\because)$$

But $(F_{m(k)})_k$ is tight so

$$F_{m(k)} \xrightarrow{w} \tilde{F}.$$

But, then $\varphi_{F_m} \to \varphi_{\tilde{F}}$ so $\varphi_{\tilde{F}} = \varphi_F$.

Since there is a 1-1 correspordance between $\varphi_{\tilde{F}}$ and $\tilde{F}$, we have $\tilde{F} = F$, which contradicts (☺) ↯

Claim: (F_n) is tight.

Proof of the Claim: Let $\varepsilon > 0$. Since

$$\varphi_n(\theta) + \varphi_n(-\theta) = 2\int \cos(\theta x)\, dF_n(x)$$

is real $\Rightarrow$ $g(\theta) - g(-\theta)$ is real.

Since g is continuous at θ, $\exists \delta > 0$ s.t.

$$|1 - g(\theta)| < \frac{1}{4}\varepsilon \qquad \forall |\theta| < \delta.$$

Thus

$$0 \leq \delta^{-1} \int_0^\delta (2 - g(\theta) - g(-\theta))\, d\theta$$

$$< \frac{1}{2}\varepsilon.$$

Since $g = \lim \varphi_n$, the BCT ensures $\exists n_0$ s.t. $\forall n > n_0$

$$\underbrace{\delta^{-1} \int_0^\delta (2 - \varphi_n(\theta) - \varphi_n(-\theta))\, d\theta}_{(B)} \leq \varepsilon.$$

Nonetheless, by Fubini's

$$(\heartsuit) = \delta^{-1}\int_{-\delta}^{\delta}\int (1-e^{i\theta x})\, dF_n(x)\, d\theta$$

$$= \delta^{-1}\int\int_{-\delta}^{\delta}(1-e^{i\theta x})\, d\theta\, dF_n(x)$$

$\int_{-\delta}^{\delta}\sin(\theta)=0$

$$= 2\int\left(1-\frac{\sin(\delta x)}{\delta x}\right) dF_n(x)$$

Since $|\sin(x)| = \left|\int_0^x \cos(x)' dx\right| \leq |x|$

$\Rightarrow 1-\frac{\sin(\delta x)}{\delta x}\geq 0$ and by discarting $[-\frac{2}{\delta},\frac{2}{\delta}]$

$$\geq 2\int_{|x|>2/\delta}\left(1-\frac{1}{|\delta x|}\right) dF_n(x)$$

$$\geq \int_{|x|>2/\delta} dF_n(x)$$

$$= \mu\left([-2\delta^{-1},2\delta^{-1}]^c\right),$$

which establishes tightness. $\square$