Lecture 24

Last time

- Strong Markov Property
- Reflection Principle

Today

- Continuous artingales
- BM as a martingale

Continuous Martingales

Given a right-continuous filtration $\{\mathcal{F}_t\}$ we say that a process $\{X_t\}_{t\in\mathbb{R}}$ is a martingale if

(1) $\mathbb{E}|X_t| < \infty \quad \forall t.$

(2) $X_t \in m\mathcal{F}_t.$

(3) If $s < t \Rightarrow \mathbb{E}[X_t | \mathcal{F}_s] = X_s.$

Theorem (♡): Assume X_t is a right-continuous martingale w.r.t. to a right-continuous filtration, and T is a stopping time s.t. $\exists C > 0$ with $\mathbb{P}(T < C) = 1$. Then

$$\mathbb{E}X_T = \mathbb{E}X_0.$$

Proof: We only give a sketch of the proof:

1. Define stopping times T_k s.t. T_k takes values in a discrete set Δ_k and $T_k \downarrow T$.
2. Restricted to Δ_k, the process is a discrete-time martingale. Use Optional Stopping to conclude

$$\mathbb{E} X_{T_k} = \mathbb{E} X_0 .$$

3. Since X_t is right continuous $\Rightarrow X_{T_k} \to X_T$ a.s.
4. Use the fact that T is bounded to conclude that $\{X_{T_k}\}_k$ is UI. Then, argue that $X_{T_k} \to X_T$ in $\mathcal{L}^1$. $\square$

BM as a martingale

Theorem (M): B_t is a martingale w.r.t. $\mathcal{F}_t^+$.

Proof: We focus on (8), the Markov Property implies

$$\mathbb{E}_x[B_t | \mathcal{F}_s] = \mathbb{E}_{B_s}(B_{t-s}) \overset{\text{Normality}}{=} B_s . \quad \square$$

Fact: For any $a \in \mathbb{R}$, $T_a = \inf\{t > 0 : B_t = a\} < \infty$ a.s.

Theorem (:-): If $a < x < b \Rightarrow \mathbb{P}_x(T_a < T_b) =$

$(b-x)/(b-a)$.

Proof: Let $T = T_a \wedge T_b$, by the fact $T < \infty$ a.s. Using Theorems (♡) and (⋈) we derive $x = E_x B(T \wedge t)$. Letting $t \to \infty$ and using BCT, we have

$$x = a\,\mathbb{P}_x(T_a < T_b) + b(1 - \mathbb{P}_x \mathbb{P}(T_a < T_b)),$$

the result follows by rearranging. $\square$

Theorem (⚡) $B_t^2 - t$ is a martingale.

Proof: We write $B_t^2 = (B_s + (B_t - B_s))^2$

$$\begin{aligned}
\mathbb{E}_x(B_t^2 \mid \mathcal{F}_s) &= E_x(B_s^2 + 2B_s(B_t - B_s) + (B_t - B_s)^2 \mid \mathcal{F}_s) \\
&= B_s^2 + 2B_s\,\mathbb{E}(B_t - B_s \mid \mathcal{F}_s) + \mathbb{E}_x[(B_t - B_s)^2 \mid \mathcal{F}_s] \\
&\overset{\text{Normality}}{=} B_s^2 + t - s.
\end{aligned}$$

$\square$

Theorem: Let $T = \inf\{t : B_t \notin (a,b)\}$ where $a < 0 < b$. Then, $\mathbb{E}_0 T = -ab$.

Interpration: BM grows on average like $\sqrt{t}$, since $T_{\sqrt{t}} = \inf\{s > 0 : B_s \notin (-\sqrt{t}, \sqrt{t})\}$

$$\mathbb{E}\, T_{\sqrt{t}} = t.$$

Proof: By Theorems (♡) and (⚡) we have $\mathbb{E}_0(B^2(T\wedge t)) = \mathbb{E}_0(T\wedge t)$. Taking $t\uparrow\infty$ and using MCT gives $\mathbb{E}_0 T$. Using BCT and Theorem (:-)

$$\mathbb{E}_0 B^2(T\wedge t) \to \mathbb{E}_0 B_T^2 = a^2\frac{b}{b-a} + b^2\frac{-a}{b-a} = -ab.$$

$\square$

We can also get results for exponentials

Theorem (xx) For any positive scalar $\theta > 0$, $\exp(\theta B_t - \theta^2 t/2)$ is a martingale.

Proof: Again we add and substract

$$\mathbb{E}_x[\exp(\theta B_t)|\mathcal{F}_s] = \exp(\theta B_s)\mathbb{E}[\exp(\theta(B_t - B_s))|\mathcal{F}_s]$$

$$= \exp(\theta B_s)\exp(\theta^2(t-s)/2)$$

where the last equality follows since $\exp(\theta(B_t - B_s))$ is independent of B_s and $B_t - B_s \sim N(0, t-s)$. Rearranging gives the result.

$\square$

Using this result we give a closed form expression $\mathbb{E}\exp(-\lambda T_a)$ with

$T_a = \inf\{t > 0 : B_t = a\}$.

Theorem: For every $\lambda > 0$, we have
$$\mathbb{E}_0 \exp(-\lambda T_a) = \exp(-a\sqrt{2\lambda}).$$

Proof: By Theorems (♡) and (😵), we have $1 = \mathbb{E}_0 \exp(\theta B_{T\wedge t} - \theta^2 T_a \wedge t/2)$, taking $\theta = \sqrt{2\lambda}$ and letting $t \uparrow \infty$ via BCT gives
$$1 = \exp(a\sqrt{2\lambda})\, \mathbb{E}_0 \exp(-\lambda T_a),$$
which is equivalent to the desired result. $\square$

In turn, we can apply this recipe to polynomials satisfying the heat equation

Theorem: If $u(t,x)$ is a polynomial in t and x s.t.
$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} = 0$$
$\Rightarrow$ $u(t, B_t)$ is a martingale.

We refer the interested reader to Theorem† 7.5.8 of Durrett. In summary this Theorem applies to all the martingales we have seen today (even the exponential one why?), and can get it us even further results.

Recap of the Class

- Asymptotics in distribution
  - Central Limit Theorem.
  - Law of rare events.
- Martingales
- Markov Chain
- Brownian Motion.

See you next week for the project presentations!