Lecture 23

Last time
- ▷ Markov Property
- ▷ consequences

Today
- ▷ Strong Markov Property
- ▷ Consequences

Question: Imagine we start $B_0 = 0$, then what is the less likely point $t \in [0,1]$ to be the last point in $[0,1]$ $B_t = 0$? What's the distribution of $L = \sup\{t : B_t = 0\}$?

The answer to the first question is $1/2$!

Stopping times

We say that a Filtration $\{\mathcal{F}_t\}$ is **right continuous** if

$$\bigcap_{s>t} \mathcal{F}_s = \mathcal{F}_t.$$

The reason we like right continuous filtrations is that they make infenitesimals into the future negligable.

Def: We say that a random variable

S in $[0,\infty]$ is a stopping time w.r.t. a filtrations $\{\mathcal{F}_t\}$ if $\{S<t\} \in \mathcal{F}_t$ $\forall t$.

This is the same as $\{S \le t\} \in \mathcal{F}_t$ if $\{\mathcal{F}_t\}$ is r.c. Why?

We need to understand what type of things are stopping times?

Q: Say that $G \subseteq \mathbb{R}$ is an open set or a closed set. Is $T_G = \inf\{t : B_t \in G\}$ a stopping time? Yes!

Theorem: If G is an open set, then T_G is a stopping time.

Proof: Since G is open and $t \to B_t$ is continuos, then

$$\{T<t\} = \bigcup_{\substack{q<t \\ q \in \mathbb{Q}}} \{B_q \in G\},$$

thus we conclude that $\{T<t\} \in \mathcal{F}_t$. $\square$

Theorem: Suppose that T_n is a sequence of stopping times. If either

$T_n \downarrow T$ or $T_n \uparrow T$.

Then, T is a stopping time. $\dashv$

Proof: It suffices to note that

$\{T<t\} = \bigcup_n \{T_n < t\}$ and $\{T \le t\} = \bigcap \{T_n \le t\}$ $\dashv$

Theorem: If G is closed, then T_G is a stopping time.

Proof: Let $B(x,r) = \{y : |y-x|<r\}$, let $G_n = \bigcup_{x\in K} B(x, \frac{1}{n})$ and let $T_n = \inf\{t \ge 0 : B_t \in G_n\}$. Since G_n is open $\Rightarrow$ T_n is a stopping time. Next we show that $T_n \uparrow T$. Notice that by construction $T_n \le T$ and $T_n \uparrow t^*$ for some $t^* < \infty$. Since $B_{T_n} \in \overline{G_n}$ $\forall n$ $\Rightarrow$ $B_{T_n} \to B_{t^*} \in G$ and so $t^* \ge T$ $\Rightarrow$ $\lim T_n = T$. $\square$

Strong Markov Property

We develop an analogue of the SMP. We now define the random shift operator. Given a nonnegative rV S in $[0,\infty]$, define

$$(\theta_S(\omega))(t) = \begin{cases} \omega(S(\omega)+t) & \text{on } \{S<\infty\} \\ \Delta & \text{on } \{S=\infty\} \end{cases}$$

Extra symbol.

We also define the information known at time S:

$$\mathcal{F}_S = \{A : A \cap \{S \le t\} \in \mathcal{F}_t \text{ for all } t \ge 0\}.$$

Proposition: If $S \le T$ are stopping times $\Rightarrow \mathcal{F}_S \subseteq \mathcal{F}_T$. $\dashv$

Proposition: If $T_n \downarrow T$ are stopping times $\Rightarrow \mathcal{F}_T = \bigcap \mathcal{F}_{T_n}$. $\dashv$

Exercise: Prove these two facts!

Theorem (Strong Markov Property). Let $(s, \omega) \to Y_s(\omega)$ be bounded and $\mathbb{R} \times \Omega$ measurable. If S is a stopping time, then for all $x \in \mathbb{R}$

$$\mathbb{E}_x [Y_S \circ \theta_S \mid \mathcal{F}_S] = \mathbb{E}_{B_S} Y_S \quad \text{on } \{S < \infty\}. \dashv$$

Function $\varphi(x, t) = \mathbb{E}_x Y_t$ evaluated at $x = B_S$ and $t = S$.

The proof of this result is similar to the one we covered in Lecture 16 (albeit much more technical), see Theorem

7.3.9. in Durrett.

Reflection Principle

Let $a>0$ and $T_a = \inf\{t : B_t = a\}$

Theorem:

$$\mathbb{P}_0(T_a < t) = 2\,\mathbb{P}_0(B_t \geq a).$$

Proof: We shall see that

$$\mathbb{P}_0(T_a < t,\ B_t > a) = \frac{1}{2}\,\mathbb{P}_0(T_a < t), \quad (\heartsuit)$$

which right away implies:

$$\mathbb{P}_0(T_a < t) = 2\,\mathbb{P}_0(B_t \geq a)$$

since $\{T_a < t\} \supseteq \{B_t > a\}$.

We focus on $(\heartsuit)$. We will use the SMP, define

$$Y_s(\omega) = \begin{cases} 1 & \text{if } s<t,\ \omega(t-s) > a, \\ 0 & \text{otherwise.} \end{cases}$$

If we let $S = \inf\{s < t : B_s = a\}$ with $\inf \emptyset = \infty$, then

$$Y_S(\theta_S(\omega)) = \begin{cases} 1 & \text{if } S<t,\ B_t > a \\ 0 & \text{otherwise.} \end{cases}$$

So SMP gives

$$\mathbb{E}_0(Y_S \circ \theta_S \mid \mathcal{F}_S) = \mathbb{E}_{B_S}(Y_S) = \mathbb{E}_a(Y_S)$$

on $\{S < \infty\} = \{T_a < t\}$

$$= \frac{1}{2}$$

← Gaussian centered at a.

Taking expectations

$$\mathbb{P}_0(T_a < t,\, B_t \geq a) = \mathbb{E}_0[(Y_S \circ \theta_S)\, \mathbb{1}_{\{S<\infty\}}]$$

$$= \mathbb{E}_0\left[\mathbb{E}_0[Y_S \circ \theta_S \mid \mathcal{F}_S]\, \mathbb{1}_{\{S<\infty\}}\right]$$

$$= \frac{1}{2}\mathbb{E}_0[\mathbb{1}\{T_a < t\}]$$

$$= \frac{1}{2}\mathbb{P}(T_a < t). \qquad \square$$

But this means that we have a closed form for the dist. of T_a:

$$\mathbb{P}_0(T_a \leq t) = 2\,\mathbb{P}_0(B_t \geq a) = \frac{2}{\sqrt{2\pi t}} \int_a^\infty \exp(-x^2/2t)\,dx.$$

Recall we were interested in $L = \inf\{t \leq 1 : B_t = 0\}$.

Then,

$$\mathbb{P}_0(L \leq t) \overset{\text{HW 5}}{=} \int_{-\infty}^{\infty} p_t(0,y)\,\mathbb{P}_y(T_0 > 1-t)\,dy$$

$$= 2\int_0^\infty (2\pi t)^{-1/2} \exp(-y^2/2s)$$

$$\int_{1-s}^\infty (2\pi u)^{-1/2} \exp(-x^2/2t)\,dx$$

$\vdots$ ← Change of variables and simplifications

$$= \frac{1}{\pi}\int_0^t (s(1-s))^{-1/2}\,ds = \frac{2}{\pi}\arcsin(\sqrt{s})$$

Thus, the density of L is $\frac{1}{\pi}(s(1-s))^{-1/2}$ which looks like