Lecture 22

Wed Apr/10/2024

Last time

- Existence

Today

- The Markov Property
- Consequences

The Markov Property

Now that we are convinced of the existance of B_t, we can study its properties. We didn't cover it, but it is relatively easy to show that B_t is nowhere differentiable (Theorem 7.16 Durrett).

Today, we study the Markov Property, which intuitively reads: "For $s \geq 0$ $B(s+t) - B(s)$ is a Brownian motion starting at $B(s)$ and ind. of what happened in the past."

We consider two slightly different filtrations:

$$\mathcal{F}_t^0 = \sigma(\{B_s : s \leq t\})$$

$$\mathcal{F}_t^+ = \bigcap_{u > t} \mathcal{F}_u^0 \qquad \text{(germ field)}$$

Notice that for $t < s$, $\mathcal{F}_t^0 \subseteq \mathcal{F}_t^+ \subseteq \mathcal{F}_s$.
Indeed $\mathcal{F}_t^+$ is right continuous, i.e.

$$\bigcap_{s>t} \mathcal{F}_s^+ = \bigcap_{s>t} \bigcap_{u>s} \mathcal{F}_u^0 = \bigcap_{u>t} \mathcal{F}_u^0 = \mathcal{F}_t^+.$$

Intuitively, $\mathcal{F}_s^+$ sees infinitesimally into the future.

Example: The random variable

$$\limsup_{s \downarrow t} \frac{B_s - B_t}{s - t}$$

is measurable w.r.t. $\mathcal{F}_t^+$, but not w.r.t. $\mathcal{F}_t^0$. ⊣

We shall see that the distinction between the two is rather negligible.

For any $x \in \mathbb{R}$, we let $\mathbb{P}_x$ be the prob. dist. of Brownian Motion with $B_0 = x$.

Recall that we constructed BM via $B_t(\omega) = \omega(t)$.

Def: For any $s \geq 0$, define the shift operator $\theta_s : \Omega \to \Omega$ by

$$\theta_s(\omega)(t) = \omega(s+t).$$

⊣

Theorem (Markov property): If $s \geq 0$ is bounded and $\mathcal{F}$ measurable, then for all $x \in \mathbb{R}$

$$\mathbb{E}_x(Y \circ \theta_s \mid \mathcal{F}_s^+) = \mathbb{E}_{B_s} Y$$

where the right-hand side is the function $\varphi(x) = \mathbb{E}_x Y$ evaluated at B_s. $\dashv$

The proof is very similar to our proof of the Markov Property in Lecture 15, where we established the result for simple functions and leveraged the π-λ Theorem and the Monotone class Theorem to extend. We defer the interested student to (Theorem 7.2.1 in Durrett).

Instead we focus on consequences.

Consequences

First we show a couple of intermediate results.

Theorem: If Z is measurable w.r.t. $\mathcal{F}$ and bounded then $\forall s \geq 0$ and $x \in \mathbb{R}^d$

$$\mathbb{E}_x(Z \mid \mathcal{F}_s^+) = \mathbb{E}_x(Z \mid \mathcal{F}_s^\circ).$$

Proof: Thanks to the Monotone Class Theorem it suffices to show this for

$$Z = \prod_{m=1}^{n} f_m(B_{t_m})$$

with f_m measurable and $0 < t_1 < \dots < t_n$.

Notice that we can split Z into two: let $\Delta = \{m \leq n : t_m \leq s\}$

$$Z = \underbrace{\left(\prod_{m \in \Delta} f_m(B_{t_m})\right)}_{X \in m\mathcal{F}_s^0} \underbrace{\left(\prod_{m \in \Delta^c} f_m(B_{t_m})\right)}_{W}$$

further we can write $W = Y \circ \theta_s$ (simply take $Y = \prod_{m \in \Delta^c} f_m(B_{t_m - s})$). Thus,

$$\mathbb{E}[Z \mid \mathcal{F}_s^+] = \mathbb{E}[X(Y \circ \theta_s) \mid \mathcal{F}_s^+]$$

$$= X \, \mathbb{E}[Y \circ \theta_s \mid \mathcal{F}_s^+]$$

Markov $\rightarrow = X \, \mathbb{E}_{B_s} Y = (\star)$

Furter notice that $\mathbb{E}_{B_s} Y \in \mathcal{F}_s^0$.

Recall that if $\mathcal{F} \subseteq \mathcal{G}$ and $\mathbb{E}[Q \mid \mathcal{G}] \in m\mathcal{F}$

$\Rightarrow \mathbb{E}[Q \mid \mathcal{G}] = \mathbb{E}[Q \mid \mathcal{F}]$ (Lecture 5)

Thus

$$(\star) = X \, \mathbb{E}[Y \circ \theta_s \mid \mathcal{F}_s^0]$$

$$= \mathbb{E}[Z \mid \mathcal{F}_s^0].$$

$\square$

This theorem implies that if $Z \in m\mathcal{F}_s^+$, then

$Z = \mathbb{E}_x[Z \mid \mathcal{F}_s^\circ] \in \mathcal{F}_s^\circ$, so $\mathcal{F}_s^+$ and $\mathcal{F}_s^\circ$ are the same up to zero measure sets.

This has beautiful consequences.

Theorem (Blumenthal's 0-1 law) If $A \in \mathcal{F}_0^+$, then $\forall x \in \mathbb{R}$,

$$\mathbb{P}_x(A) \in \{0, 1\}.$$

Proof: By the previous theorem

$$\mathbb{1}_A = \mathbb{E}_x[\mathbb{1}_A \mid \mathcal{F}_0^+] = \mathbb{E}_x[\mathbb{1}_A \mid \mathcal{F}_0^\circ] = \mathbb{P}_x(A)$$

almost surely. $\square$

This is particularly useful to undestand the behavior of BM locally.

Corollary: Consider $\tau_+ = \inf\{t > 0 : B_t > 0\}$, then $\mathbb{P}_0(\tau_+ = 0) = 1$.

Proof: Since the normal dist is symmetric around 0

$$\mathbb{P}_0(\tau_+ \leq t) \geq \mathbb{P}_0(B_t > 0) = 1/2.$$

Moreover, letting $t \downarrow 0$

$$\mathbb{P}_0(\tau_+ = 0) = \lim_{t \downarrow 0} \mathbb{P}(\tau_+ \leq t) \geq 1/2.$$

So by Blumenthal's 0-1 law, $\mathbb{P}_0(\tau_+ = 0) = 1.$ $\square$

Notice that the same conclusion holds for $\tau_- = \inf\{t>0 : B_t < 0\}$. Further B_t is continuous so we can easily derive:

Corollary: Consider $\tau_0 = \inf\{t>0 : B_t = 0\}$, then $\mathbb{P}_0(\tau_0 = 0) = 1$. $\dashv$

Blumenthal's 0-1 law allows us to understand questions as $t \to 0$, but how about when $t \uparrow \infty$? In turn, we can reverse BM and understand asymptopia via $t \downarrow 0$.

Theorem: If B_t is a BM starting at 0, then so is the process defined by $X_0 = 0$ and $X_t = t B_{1/t}$ as $t \to \infty$.

Next class we will prove this result, but for now let us cover a relevant consequence. Let

$$\mathcal{F}_t' = \sigma(B_s : s > t)$$ The future at t.

$$\mathcal{T} = \bigcap_{t \geq 0} \mathcal{F}_t'$$ Tail σ-algebra.

Theorem: Let $A \in \mathcal{J}$, then either $\mathbb{P}_x(A) = 0 \ \forall x$ or $\mathbb{P}_x(A) = 1 \ \forall x$.

TO BE CONTINUED . . .