Lecture 21

Mon Apr/08/2024

Last time

- Brownian Motion
- Formal construction

Today

- Existence

Last time we closed with

Theorem (♡): μ assigns probability one to paths $\omega : \mathbb{Q}_2 \longrightarrow \mathbb{R}$ that are uniformly continuous in $\mathbb{Q}_2$. ⊣

The proof of this result follows easily from:

Process defined on $(\Omega_q, \mathcal{F}_q)$

Theorem (•): Suppose that $\mathbb{E}|X_s - X_t|^\beta \leq K|t-s|^{1+\alpha}$ where $\alpha, \beta > 0$. Then if $\gamma < \alpha/\beta$ then with probability one $\exists\, C(\omega)$ so that

$$|X_q - X_r| \leq C|q-r|^\gamma \quad \forall q, r \in \mathbb{Q}_2.$$

Proof of Theorem (♡):

Notice that by our construction

$$\mathbb{E}|B_t - B_s|^4 = \mathbb{E}|B_{t-s}|^4 = \mathbb{E}|(t-s)^{1/2} B_1|^4$$

$$= (t-s)^2\, \mathbb{E}\, \underbrace{B_1}_{N(0,1)}{}^4 = 3(t-s)^2.$$

Thus invoking Theorem (·) yields a.s. $\exists$ C s.t.

$$|B_t - B_s| \le C\,|t-s|^{1/2} \quad \forall t, s \in \mathbb{Q}_2,$$

which immediately implies uniform continuity. $\dashv$

Proof of Theorem (·):

Note that it suffices to show that

$$|X_q - X_r| \le A\,|q-r|^{\gamma} \quad \forall q, r \in \mathbb{Q}_2 \text{ s.t. } |q-r| \le \delta. \qquad (\infty)$$

Real valued r.v. $A(\omega)$, $\delta(\omega) > 0$

If (∞) holds then $\forall s, t \in \mathbb{Q}_2$ we can find $s = s_0 < \dots < s_n = t$ s.t. $|s_i - s_{i-1}| \le \delta$ and

$$\begin{aligned}
|X_s - X_t| &\le |X_s - X_{s_1}| + |X_{s_2} - X_{s_1}| + \dots + |X_t - X_{s_n}| \\
&\le A\left(|s-s_1|^{\gamma} + \dots + |X_t - s_n|^{\gamma}\right) \\
&= A\left(\frac{|s-s_1|^{\gamma}}{|s-t|^{\gamma}} + \dots + \frac{|X_t - s_n|^{\gamma}}{|s-t|^{\gamma}}\right)|s-t|^{\gamma} \\
&\le A\left(\sum_{i=1}^{\lceil \delta^{-1} \rceil} 1\right)|s-t|^{\gamma} \\
&\le \underbrace{A\lceil \delta^{-1} \rceil}_{C(\omega)}\,|s-t|^{\gamma}.
\end{aligned}$$

Thus, we focus on proving (∞).

Let

$$G_n = \left\{ \begin{array}{l} |X(m/2^n) - X(m-1/2^n)| < 2^{-\gamma n} \\ \text{for all } 1 \le m \le 2^n \end{array} \right\}$$

By Markov's ineq:

$$\mathbb{P}\left[\overbrace{|X(m/2^n) - X(m-1/2^n)|}^{Y_m} \ge 2^{-\gamma n}\right] \le \left[\mathbb{E}\, |Y_m|^\beta\right] 2^{\gamma n \beta}$$

$$= K\, 2^{-n(1+\alpha-\gamma\beta)}$$

Thus, taking union bound

$$\mathbb{P}(G_n^c) \le 2^n K\, 2^{-n(1+\alpha-\gamma\beta)} = K 2^{-n(\alpha-\gamma\beta)}$$

The proof follows from the following Lemma:

Lemma (⚡) On $H_N = \bigcap_{n=N}^{\infty} G_n$, we have

$$|X_q - X_r| \le \frac{4}{1-2^{-\gamma}} |q-r|^\gamma \quad \forall\, q, r \in Q_2$$

s.t. $|q-r| < 2^{-N}$. ⊣

We will come back to the proof of this Lemma, but for now note that

$$P(H_N^c) \le \sum_{n=N}^{\infty} P(G_n^c) \le K \sum_{n=N}^{\infty} 2^{-n(\alpha-\gamma\beta)}$$

$$= \frac{K}{1-2^{-(\alpha-\gamma\beta)}} \, 2^{-N(\alpha-\gamma\beta)} .$$

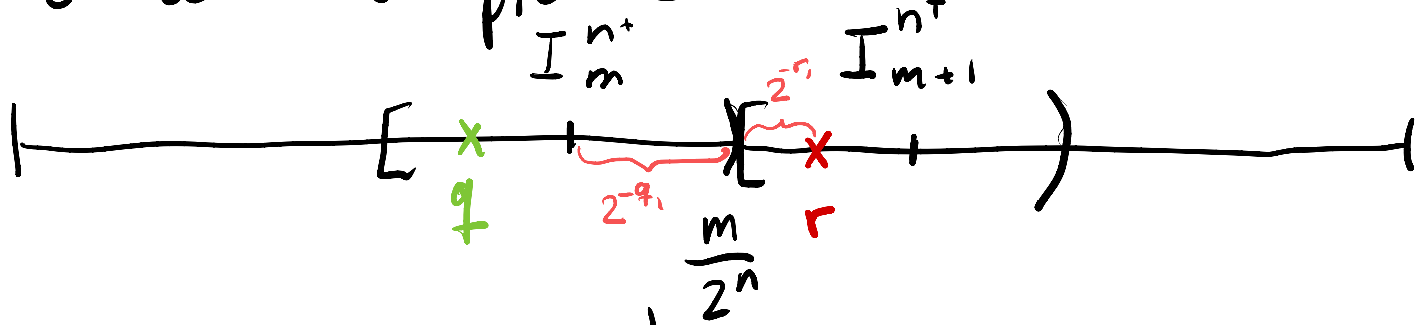
Since $\gamma < \frac{\alpha}{\beta} \Rightarrow \sum_{N=1}^{\infty} \mathbb{P}(H_N^c) < \infty$. By Borell Cantelli, H_N^c only holds for finite N, so (∞) holds, concluding the proof of Theorem $(\cdot)$. $\square$

Proof of Lemma (↯): Take $r, q \in \mathbb{Q}_2$ with $0 < r - q < 2^{-N}$. Define

$$I_m^n = [m-1/2^n, \, m/2^n).$$

Let $n^+ = \max\{ n : \exists m \text{ s.t } q \in I_m^n \text{ and } r \in I_{m+1}^n \}$.

Let's see a picture:



Then we can write

$$r = m2^{-n^+} + 2^{-r_1} + \dots + 2^{-r_\ell}$$

$$q = m2^{-n^+} - 2^{q_1} - \dots - 2^{q_k}$$

where $N<q_1<\dots<q_k$ and $N<r_1<\dots<r_\ell$.

Then,

$$|X_q - X_r| \le |X_q - X(m/2^{-n^+})| + |X(m/2^{-n^+}) - X_r|$$

$N<q_i$
$N<r_j$

$$\le \sum_{j=1}^{k} 2^{-q_j\gamma} + \sum_{i=1}^{\ell} 2^{-r_i\gamma}$$

$2^{-q_i}<2^{-n^+}$
$2^{-r_j}<2^{-n^+}$

$$\le 2\sum_{n=n^+}^{\infty} 2^{n\gamma}$$

$$\le \frac{2}{1-2^{\gamma}} 2^{-n^+\gamma}$$

Since n^+ is a max
$2^{-(n^++1)} \le |q-r|$

$$\le \frac{4}{1-2^{\gamma}} |q-r|^{\gamma}$$

$\square$