Lecture 20 Wed Apr/3/2024

Last time

- Aperiodicity
- Convergence Thm.

Last class on MC.

Today

- Brownian Motion
- Existance

Brownian Motion

So far in this class we have study several general objects that exhibit convergence under mild assumptions (Martingales and Markov Chains).
For the remainder of this class we will focus on a single stochastic process that does not converge but it is in a certain sense universal and so it is central in multiple areas of Applied Math: the Brownian Motion.

A bit of history:

- Robert Brown 1827 (Described the physical phenomenon for pollen suspended in water)

Recommedation Veritasium video.

- Louis Bachelier 1900 (Described the stochastic process in his PhD thesis in finance)

Founder of Math Finance

- Albert Einstein 1905 (Models the movement of pollen as a the result of it being hit by individual water particles).

Evidence for the existence of atoms

- Jean Perrin 1908 (Experimentally verifies Einstein's model)

1926 Nobel Prize in Physics.

The formal construction

Unlike all the processes we have seen the Brownian Motion is indexed by $\mathbb{R}_+$: B_t with $t \geq 0$ is a collection of random variables defined on a prob. space $(\Omega, \mathcal{F}, \mathbb{P})$ s.t:

1) If $t_0 < t_1 < \dots < t_n$ then

$$B(t_0),\ B(t_1) - B(t_0),\ \dots,\ B(t_n) - B(t_{n-1})$$

are independent.

2) If $t > s$, $B(t) - B(s) \sim N(0, t-s)$

3) With probability one, $t \to B_t$ is con

tinuous. $t \mapsto B_t(\omega)$ for almost all $\omega \in \Omega$. $\dashv$

A couple of important facts:

Fact 1: $\{B_t - B_0,\ t \geq 0\}$ is independent of B_0 and has the same distribution as the BM with $B_0 = 0$. $\dashv$

Fact 2: If $B_0 = 0$, then for any $t > 0$

$$\{B_{st} \geq 0\}_{s \geq 0} \overset{d}{=} \{t^{1/2} B_s\}_{s \geq 0}. \quad \dashv$$

Formally this just means that $\forall\, 0 < s_1 < \dots < s_n$

$$(B_{s_1 t}, \dots, B_{s_n t}) \overset{d}{=} t^{1/2} (B_{s_1}, \dots, B_{s_n}).$$

Exercise: Prove these two facts formally.

Existence

Thanks to the two facts we can focus on proving existence of B_t for $t \in [0,1] =: I$ and $B_0 = 0$.

Intuitively we would like to have

$$\Omega = \{\text{Functions } \omega : I \to \mathbb{R}\}$$

$$\mathcal{F} = \sigma\big(\{\omega : \omega(t_i) \in A_i \text{ for } 1 \leq i \leq n.\}\big)$$

Can only see countable points

Unfortunately the event {w is continuous} is not measurable. So we need to take a bit of a detour and consider

$$\mathbb{Q}_2 = \{ m 2^{-n} : n \in \mathbb{N}, \ m \leq 2^n \}$$

$$\Omega_q = \{ \text{Functions } w : \mathbb{Q}_2 \to \mathbb{R} \}$$

$$\mathcal{F}_q = \sigma(\{ w : w(t_i) \in A_i \text{ for } 1 \leq i \leq m \})$$

Note that if w continuous then there is a unique extension $\bar{w} : I \to \mathbb{R}$ of w that is continuous ($\bar{w}(t) := \psi(w)(t) = \lim_{\substack{s \in \mathbb{Q}_2 \\ s \to t}} w(s)$).

Thus, if we endow $(\Omega_q, \mathcal{F}_q)$ with a prob. measure μ satisfying (1), (2), and (3) (restricted to $\mathbb{Q}_2$) then there is a natural pass from $(\Omega_q, \mathcal{F}_q, \mu)$ to $(\Omega, \mathcal{F}, \mathbb{P})$ via ψ with

$$\mathbb{P} = \mu \circ \psi^{-1}.$$

Thus, we focus on the construction of μ on $(\Omega_q, \mathcal{F}_q)$. We define μ by first considering its marginals into finitely many times. For any $t_1 < \ldots < t_n \in Q_2$, define:

$x_0 = 0$

$$\mu_{t_1 \ldots t_n}(A_1, \ldots, A_n) = \int_{A_1} dx_1 \ldots \int_{A_n} dx_n \prod_m f_{t_m - t_{m-1}}(x_m - x_{m-1})$$

$(\underbrace{Z_{t_1}}_{B_{t_1}}, \underbrace{Z_{t_1} + Z_{t_2 - t_1}}_{B_{t_2}}, \ldots, Z_{t_1} + \ldots + Z_{t_n - t_{n-1}})$ Density of $N(0, t_k - t_{k-1})$

We would like to extend the marginals to a μ defined on $(\Omega_q, \mathcal{F}_q)$, we can do so via Kolmogorov's Extension Theorem, provided

$$\mu_{t_1 \ldots t_{k-1} t_{k+1} \ldots t_n}(A_1 \times \ldots \times A_{k-1} \times A_k \times \ldots \times A_n)$$

(*) $$= \mu_{t_1 \ldots t_n}(A_1 \times \ldots \times A_{k-1} \times \mathbb{R} \times A_k \times \ldots \times A_n).$$

Exercise: Prove that (*) holds.

Let μ be the extension given by good old Kolmogorov. We define $B_t \sim \mu$ via $B_t(\omega) = \omega(t)$.

Exercise: Show that B_t satisfies (1) and (2) provided that we restrict $t \in Q_2$.

The difficult part now is to show

that $t \mapsto B_t$ is a.s. continuous.

Theorem (♡): μ assigns probability one to paths $\omega : \mathbb{Q}_2 \to \mathbb{R}$ that are uniformly continuous in $\mathbb{Q}_2$.

The proof of this result follows easily from:

Process defined on $(\Omega_q, \mathcal{F}_q)$

Theorem (•): Suppose that $\mathbb{E}|X_s - X_t|^\beta \leq K|t-s|^{1+\alpha}$ where $\alpha, \beta > 0$. Then if $\gamma < \alpha/\beta$ then with probability one $\exists\, C(\omega)$ so that

$$|X_q - X_r| \leq C|q-r|^\gamma \quad \forall q, r \in \mathbb{Q}_2.$$

⊣

Proof of Theorem (♡):

Notice that by our construction

$$\mathbb{E}|B_t - B_s|^4 = \mathbb{E}|B_{t-s}|^4 = \mathbb{E}|(t-s)^{1/2} B_1|^4$$
$$= (t-s)^2\, \mathbb{E} B_1^4 = 3(t-s)^2.$$

$N(0,1)$

Thus invoking Theorem (•) yields a.s. $\exists\, C$ s.t.

$$|B_t - B_s| \leq C|t-s|^{1/2} \quad \forall t, s \in \mathbb{Q}_2,$$

which immediately implies uniform continuity. $\dashv$

Proof of Theorem (·):

Note that it suffices to show that

$$|X_q - X_r| \le A\,|q-r|^{\gamma} \quad \forall q,r \in Q_2 \text{ s.t. } |q-r| \le \delta. \qquad (\infty)$$

Real valued r.v. $A(\omega)$, $\delta(\omega) > 0$

If (∞) holds then $\forall s,t \in Q_2$ we can find $s = s_0 < \dots < s_n = t$ s.t. $|s_i - s_{i-1}| \le \delta$ and

$$\begin{aligned}
|X_s - X_t| &\le |X_s - X_{s_1}| + |X_{s_2} - X_{s_1}| + \dots + |X_t - X_{s_n}| \\
&\le A\left(|s-s_1|^{\gamma} + \dots + |x_t - s_n|^{\gamma}\right) \\
&= A\left(\frac{|s-s_1|^{\gamma}}{|s-t|^{\gamma}} + \dots + \frac{|x_t - s_n|^{\gamma}}{|s-t|^{\gamma}}\right)|s-t|^{\gamma} \\
&\le A\left(\sum_{i=1}^{\lceil \delta^{-1} \rceil} 1\right)|s-t|^{\gamma} \\
&\le A\lceil \delta^{-1} \rceil\,|s-t|^{\gamma}.
\end{aligned}$$

Thus, we focus on proving (∞).

Let

$$G_n = \left\{ \begin{array}{l} |X(m/2^n) - X(m-1/2^n)| < 2^{-\gamma n} \\ \text{for all } 1 \le m \le 2^n \end{array} \right\}$$

By Markov's ineq:

$$\mathbb{P}\left[\overbrace{|X(m/2^n) - X(m-1/2^n)|}^{Y_m} \geq 2^{-\gamma n}\right] \leq \left[\mathbb{E}|Y_m|^{\beta}\right] 2^{\gamma n \beta}$$

$$= K\, 2^{-n(1+\alpha-\gamma\beta)}$$

Thus, taking union bound

$$\mathbb{P}(G_n^c) \leq 2^n K 2^{-n(1+\alpha-\gamma\beta)} = K 2^{-n(\alpha-\gamma\beta)}$$

The proof follows from the following Lemma:

Lemma(3) On $H_N = \bigcap_{n=N}^{\infty} G_n$, we have

$$|X_q - X_r| \leq \frac{3}{1-2^{-\gamma}} |q-r|^{\gamma} \quad \forall\, q, r \in \mathbb{Q}_2$$

s.t. $|q-r| < 2^{-N}$.

$\dashv$

We will come back to the proof of this Lemma, but for now note that

$$\mathbb{P}(H_N^c) \leq \sum_{n=N}^{\infty} \mathbb{P}(G_n^c) \leq K \sum_{n=N}^{\infty} 2^{-n(\alpha-\gamma\beta)}$$

$$= \frac{K}{1-2^{-(\alpha-\gamma\beta)}} 2^{-N(\alpha-\gamma\beta)}.$$

Since $\gamma < \frac{\alpha}{\beta} \Rightarrow \sum_{N=1}^{\infty} \mathbb{P}(H_N^c) < \infty$. By

Borell Cantelli, H_N^c only holds for finite N, so (∞) holds, concluding the proof of Theorem $(\cdot)$.

$\square$

NEXT CLASS WE WILL GO BACK TO THE PROOF OF LEMMA (⚡).