Lecture 2 — Wed Jan/24/2024

Last time

- Logistics
- What is this course about?
- Some examples

Today

- Characteristic Functions
- Levy's inversion formula

Characteristic function

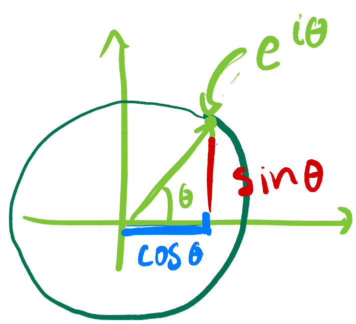
Today we will cover a different way to see random variable (r.v).

Given a r.v. X in $\mathbb{R}$ we define its ch. f. as

Fourier transform

$$\varphi_X(t) = \mathbb{E}\exp(itX) = \int \exp(itx)\, d\mu(x)$$
$$= \mathbb{E}\cos(tX) + i\,\mathbb{E}\sin(tX).$$

Recall that $e^{i\theta}$ lives in the complex circle



Ch. f.s give a way to characterize dis

tributions, unlike moment generating functions, i.e. $\mathbb{E}\exp(tX)$, they always exist.

Proposition We have that $\forall t$:

1) $\varphi(0) = 1$.
2) $\varphi(-t) = \overline{\varphi(t)}$. complex conjugate.
3) $|\varphi(t)| \leq 1$. $\leftarrow$ Well-definedness.
4) φ is uniformly continuous.
5) $\varphi_{aX+b}(t) = \exp(itb)\,\varphi_X(at)$.

Proof: 1) Trivial.

2) $\varphi(-t) = \mathbb{E}\exp(-itX) = \mathbb{E}\,\overline{\exp(itX)} = \overline{\mathbb{E}\exp(itX)}$.

3) By Jensen's

In the complex circle.

$$|\varphi(t)| \leq \mathbb{E}|\exp(itX)| = 1.$$

4) Note that for any $h \in \mathbb{R}$

$$\begin{aligned}
|\varphi(t+h) - \varphi(t)| &= |\mathbb{E}\exp(itX)(\exp(ihX) - 1)| \\
&\leq \mathbb{E}\left[|\exp(itX)|\,|\exp(ihX) - 1|\right] \\
&\leq \mathbb{E}|\exp(ihX) - 1| \\
&\to 0 \quad \text{when } h \to 0.
\end{aligned}$$

Pointwise convergence of $|e^{ihX} - 1|$ + Bounded convergence theorem since $|e^{ihX} - 1| \leq 2$.

5) $\mathbb{E}\exp(it(aX+b)) = \mathbb{E}[\exp(itaX)\exp(itb)]$
$= \exp(itb)\,\varphi(at).$ $\square$

Theorem: If X_1 and X_2 are ind. Then
$$\varphi_{X_1+X_2}(t) = \varphi_{X_1}(t)\,\varphi_{X_2}(t).$$

Proof:
$$\mathbb{E}\exp(it(X_1+X_2)) = \mathbb{E}\exp(itX_1)\exp(itX_2)$$
$$= \mathbb{E}\exp(itX_1)\,\mathbb{E}\exp(itX_2).$$
$\square$

Examples: (Coin flips)
$$\mathbb{E}e^{itX} = (e^{it}+e^{-it})/2 = \cos t.$$

(Poisson distribution) If $P(X=k)\ e^{-\lambda}\lambda^k/k!$
$$\mathbb{E}e^{itX} = \sum e^{-\lambda}\frac{\lambda^k e^{itk}}{k!} = \exp(\lambda(e^{it}-1))$$

(Normal distribution) Let $X\sim N(0,\sigma^2)$
$$\Rightarrow \varphi_X(t) = \exp(-t^2\sigma^2/2).$$
Exercise.

(Uniform distribution) Let $X\sim \mathrm{Unif}((a,b))$
$$\Rightarrow \varphi_X(t) = \frac{e^{itb}-e^{ita}}{it(b-a)}$$

Follows from $\int_a^b e^{i\lambda x}\,dx = (e^{i\lambda b}-e^{i\lambda a})/i\lambda.$

Levy's inversion formula

Characteristic functions characterize distributions.

Theorem: Let μ a probability measure, and φ its char. fun. If $a<b$, then

$$\mu(a,b) + \frac{1}{2}\mu(\{a,b\}) = \lim_{T\to\infty} \int_{-T}^{T} \frac{e^{-ita} - e^{itb}}{it}\varphi(t)\, dt.$$

Proof: Consider

$$I_t = \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it}\varphi(t)\, dt$$

$$= \int_{-T}^{T}\int \frac{e^{-ita} - e^{-itb}}{it} e^{itx}\, d\mu(x)\, dt.$$

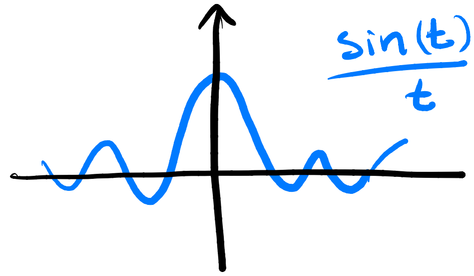
Since μ is a prob. measure and $[-T,T]$ is finite, then, by Fubini's

$$= \int\int_{-T}^{T} \frac{e^{-ita} - e^{itb}}{it} e^{itx}\, dt\, d\mu(x)$$

sin is odd and cos is even

$$= \int\int_{-T}^{T} \frac{\sin(t(x-a))}{t}\, dt - \frac{\sin(t(x-b))}{t}\, dt\, d\mu$$

Consider two cases:

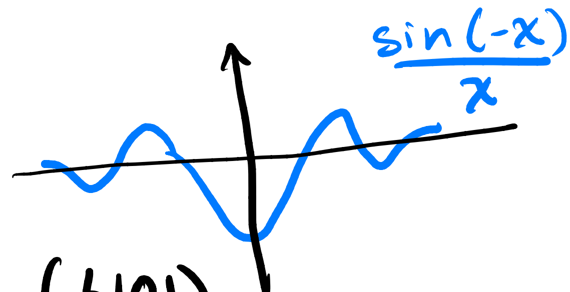


- If $(x-a) = \theta > 0$

$$\Rightarrow \int_{-T}^{T} \frac{\sin(t\theta)}{t}\, dt = 2\int_0^T \frac{\sin(t\theta)}{t}\, dt$$

$$= 2\int_0^{\theta T} \frac{\sin(s)}{s}\, ds$$

Change of variable $t = s/\theta$



- If $(x-a) = \theta < 0$

$$\Rightarrow \int_{-T}^{T} \frac{\sin(t\theta)}{t} = -\int_{-T}^{T} \frac{\sin(t|\theta|)}{t}$$

$$= -2\int_0^{|\theta| T} \frac{\sin(s)}{s}\, ds$$

Thus

$$(:) = 2\int \underbrace{\left[\operatorname{sign}(x-a)\int_0^{|x-a|T} \frac{\sin(s)}{s}\, ds - \operatorname{sign}(x-b)\int_0^{|x-b|T} \frac{\sin(s)}{s}\, ds\right]}_{w(x,\, T;\, a,\, b)} d\mu$$

Fact / Exercise: $\lim_{T\to\infty} \int_0^T \frac{\sin(s)}{s}\, ds = \frac{\pi}{2}$

complex analysis.

Using this fact it is easy to see that

$$w(x, T; a,b) \xrightarrow{T\to\infty} \begin{cases} \pi & \text{if } x \in (a,b) \\ \pi/2 & \text{if } x \in \{a,b\} \\ 0 & \text{otherwise.} \end{cases}$$

Then, since $|w(x, T; a\,b)| \leq 2 \sup_{T \in \mathbb{R}} \int_0^T \frac{\sin(s)}{s} ds < 4$ we derive by the BCT that

$$\frac{1}{2\pi} \lim_{T\to\infty} \int_{-T}^{T} 2w(x, T; ab)\, d\mu = \mu(a,b) + \frac{1}{2}\mu(\{a,b\})$$

$\square$

Remark: The Lesbegue integral is not well-defined since

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^+ dx = \int_0^\infty \left(\frac{\sin x}{x}\right)^- dx = \infty.$$

Corollary: If $\int |\varphi(\theta)| d\theta < \infty$, then X has a continuous probability density

$$f(y) = \int e^{-ity} \varphi(t)\, dt$$

Fourier transform inverse.

Proof: Notice that

$$\left| \frac{e^{-ita} - e^{-itb}}{it} \right| = \left| \int_a^b e^{-its} ds \right| \le |b-a|.$$

Then, by Dominated convergence Theorem (DMV)

$$\mu(a,b) + \frac{1}{2}\mu(\{a,b\}) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-ita} - e^{-itb}}{it} \varphi(t)\, dt$$

$$\le \frac{(b-a)}{2\pi} \int_{\mathbb{R}} |\varphi(t)|\, dt.$$

By the inequality, $\mu(\{a\}) = \mu(\{b\}) = 0$, so the above equality holds without $\mu(\{a,b\})$.

Using Fubini's

$$\mu(a,b) = \frac{1}{2\pi} \int \frac{e^{-ita} - e^{-itb}}{it} \varphi(t)\, dt$$

$$= \frac{1}{2\pi} \int \int_a^b e^{-ity} \varphi(t)\, dy\, dt$$

$$= \int_a^b \left(\frac{1}{2\pi} \int e^{-ity} \varphi(t)\, dt \right) dy,$$

Thus
$$f(y) = \frac{1}{2\pi} \int e^{-ity} \varphi(t)\, dt.$$

A simple application of DCT yields continuity of f.

$\square$

Corollary 2: If φ is real, then μ is symmetric (X and $-X$ have the same distribution).

Proof: WLOG $\mu(\{a,b\}) = 0$. Exercise HW.

$$\mu(a,b) = \lim_{T\to\infty} \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} \varphi(t)\, dt.$$

$s \leftarrow -t$
$$= \lim_{T\to\infty} \int_{T}^{-T} \frac{e^{isa} - e^{isb}}{is} \varphi(-s)\, ds$$

$\varphi(-s) = \overline{\varphi(s)}$.

$$= \lim_{T\to\infty} \int_{-T}^{T} \frac{e^{isb} - e^{isa}}{is} \varphi(s)\, ds$$

$$= \mu(-b,-a)$$

$\square$

Corollary 3: Let $X \sim N(0,\sigma_x^2)$ and $Y \sim N(0,\sigma_y^2)$ independents.

Then $X+Y \sim N(0, \sigma_X^2+\sigma_Y^2)$.

Proof:

$$f_{X+Y}(s) = \frac{1}{2\pi}\int e^{-its}\,\varphi_{X+Y}(t)\,dt$$

$$= \frac{1}{2\pi}\int e^{-its}\,\varphi_X(t)\,\varphi_Y(t)\,dt$$

$$= \frac{1}{2\pi}\int e^{-it}\,e^{-t^2\sigma_X^2/2}\,e^{-t^2\sigma_Y^2/2}\,dt$$

$$= \frac{1}{2\pi}\int e^{-it}\,e^{-t^2(\sigma_X^2+\sigma_Y^2)/2}\,dt$$

This is the density of $N(0, \sigma_X^2+\sigma_Y^2)$. $\square$