Lecture ¹⁹ Mon Apr/01/2024 Last time Today i stationary measures ^I Aperiodicity Existence Convergence Theorem ^D Uniqueness

Aperiodricity Our goal today is to understand Existence $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 &$ $lim_{x \to \infty}$ P_{x} (X_{n} = –
n → K) If this yields ^a probability dist, we can ron Markov Chains for a while to sample! in Markov Chairs for a while to s
Notice that if y is transcient = \Rightarrow this If this yields
un Markov
Notice that
Limit is zero.
A natural a ^A natural question is when is that the limit exists ? Example: Consider the chain $rac{9cosh\theta}{\cosh\theta}$
Consider $\overline{\mathbf{1}}$ \bullet \bullet \bullet S_n $\overbrace{S_1}$ $\overbrace{S_2}$ Then, $S_i = \frac{1}{D(x_i + \frac{1}{2})}$ $P(S_{2n} = S) \neq P(Y_{2n+1} = S) = 0.$ $1 = P_s(X_{2n} = s_1) \neq P(X_{2n+1} = s_1) = 0.$

We shall see that this periodic behavior We shall see that this periodic behavior
is the only thing preventing convergence. Def : For any recurrent XES, the period of x , called d_x , is the greatest Bef: For any recurrent $x \, \epsilon S$, the
of x, called d_x , is the greate
common divisor of $I_x = \sqrt{n} \, 1 : p^{(n)}(x)$ x) > 0 <u>j</u>.).
|-The previous example has a period of two Example : Consider $\bullet \longleftarrow \bullet \stackrel{S_0}{\bullet}$ 5.1 called d_x , is the
divisor of $I_x = \langle n \ge 1 :$
evrous example has e : Consider

Il and Song Song Il and Il \bullet S_{22} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot S
၁ There are two cycles including $S_0: S_0 \to S_1 \to S_2 \to S_3 \to S_0$ $S_0 \rightarrow S_1 \rightarrow S_2 \rightarrow S_3 \rightarrow S_0$ $S^0 \to S^{-1} \to S^{-5} \to S^0$ Thus, $I_{s_0} = \{3n, 4n\}$, $33 - 33$
33, 30
33, 4n), therefore $d_{s_0} =$ \mathcal{I} . .
ا We say that a chain is aperdic We say that a chain is apendic
if $d_{x} = 1$ tx. In turn, aperiodicity

Holds for everyone in an irredocible class. ↓ is a "class property." $\int_{0}^{Lemma(N)}$ (\upmu): $\downarrow \downarrow$ $\rho_{xy} > 0$ \Rightarrow $d_y = d_x$. α "class proper.

emma (M): If p_{xy}

Left as exercise. Lemma $(3):$ if $d_{x} = 1$, then, $\rho^{(m)}(x, x) > 0$ for $m \geq m_0$. Proof: We will use two claims. $\frac{\rho_{\text{rod}}}{\rho_{\text{rod}}}$ $\frac{d}{d}$: We will use two claims:
Claim: If In s.t. $m \in I_x$, $m+1 \in I_x$, t_{term} : If 3m s.t. $m \in I_x$, $m+1 \in I_x$, then
the result follows. $\overline{}$ Fact: If $gcd(\mathcal{I}_x) = 1$, then $\exists i$, . the result follows.
Fact: If $gcd(I_x) = 1$, then $\exists i, ..., i_k \in I$
and $c_1, ..., c_k \in Z$ s.t. $\sum_{i=1}^{k} c_i i_l = 1$. K the result follows.
Fact: If $gcd(I_x) =$
and $c_1, ..., c_k \in \mathbb{Z}$ s. ∟
+ 1, then $3c_{1}$.
+ $\sum_{i=1}^{k} c_{i}c_{i} = 1$ Fact from number theory that we will not from rue. Let's show that these two imply the $\boldsymbol{e} = \boldsymbol{c}$ - Let's show that these two imply the $e = c_e^{\dagger}$ and $b_e^{\dagger} = c_e^{\dagger}$ maximum then $a_1\dot{c}_1 + ... + a_k\dot{c}_k = b_1$ $i_1 i_1 + ... + a_k i_k = b_i i_1 + ... + b_k i_k + 1$. m: If $\exists m s + m \in I_x$, $m+1 \in I_x$

- result follows.
 $\pm f$: If $gcd(I_x) = 1$, then
 $c_1, ..., c_k \in \mathbb{Z}$ s.t. $\sum c_j i_j$

- rom number theory that we

show that these two imply

t. Let $a_i = c_i^{\dagger}$ and $b_i =$
 $a_i \pm c_1 + ... + a_k i_k = b_i i_1 + ... +$ $m + 1E I_{\chi}$ mel_x Then, the result follows by the claim. I

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Convergence Theorem

\nWe are now ready to prove the main result belong

\nTheorem: Suppose that a MC with transition prob. p is irreducible and operation. Theorem

\nAstribubron T. Then, for any
$$
x \in S
$$

\n $\mathbb{P}_x(X_n = .) \xrightarrow{\omega} T$

\nProof: Let $S^2 = S \times S$ and define the chain given by $p((x_i, y_i), (x_z, y_z)) = p(x_i, x_z) p(y_i, y_z)$.

\n p First, we note that \overline{p} is irreducible.

\nTo see thus, role that since p is irreducible.

\nTherefore $\Rightarrow \exists k$ and L s.t. $p^{(k)}(x_i, x_k)$

\nso and $p^{(k)}(y_1, y_1) > 0$. By Lemma (3), and $p^{(k)}(x_1, x_2) > 0$ and $p^{(k)}(y_1, y_1) > 0$.

\n $\Rightarrow \overline{p}^{(k+k+M)}(x_k, x_k) > 0$ and $p^{(k+M)}(y_k, y_k) = 0$.

\nSo.

↳ Second $\frac{d}{ds}$ we note that $\overline{\pi}(a,b))$ = $\overline{\pi}(a)\pi(b)$ defines a stationary distribution (since both components are ind.) and moreover refines a sportary distribution (simulation)
both components are ind.) and moreover \bar{p} makes all states s² recurrent.
This follows from the following Lemma: If there is a in components are that, and moreoner
- makes all states s² recurrent.
- his follows from the following
- station, then all states y s.t. $\pi(y)$. Lemma: If there is a stationary dis
tribution, then all states y s.t. $\pi(y)$ 20 tribution, then all states y s.t. $\pi(y)$ to Proof of Lemma: Note that stationarity Proof of Lemma: Note H
 $\omega = \sum_{n=1}^{\infty} \pi(y) \stackrel{\star}{=} \sum_{n=1}^{\infty} \sum_{\chi} \pi(\chi) p^{(n)}(x)$ $\sum_{n=1}^{\infty} \pi(y) = \sum_{n=1}^{\infty} \sum_{\chi} \pi(x) p^{(n)}(x, y)$ Fubini's \Rightarrow = $\frac{\pi i}{\pi} \pi(x) \sum_{n=0}^{\infty} p^{(n)}(x,y)$ Fubini's \Rightarrow = $\sum_{x} \pi(x) \sum_{n=0}^{\infty} P^{n}$
Formula for $\sum_{x} \pi(x) \rho_{xy}$ Fubini's \Rightarrow = $\sum_{x} \pi(x) \sum_{n=0}^{\infty} P^{(n)}$

Fubini's \Rightarrow = $\sum_{x} \pi(x) \sum_{n=0}^{\infty} P^{(n)}$
 $E_x M(y)$ $\frac{\gamma}{\gamma}$ + (
 $\frac{1}{1-\beta}$ π is adist $3 \leq \frac{1}{2}$ π is adist $\approx \frac{1}{1-\rho_{yy}}$ So we conclude that $\rho_{yy} = 1$. \Box D Let $(X_{n}, Y_{n}) \sim \overline{P}$, let let $T =$ $in\frac{1}{2}$ $\left\{ n\ge1$ 1 $X_{n} = Y_{n}\right\}$. Note that for any fixed x we have

$$
T_x = m_x^2 \int nz_1 x_2 x_3 dx_0
$$
 a.s. since (x,x)
\nis required:
\n $\int x_n$ and y_n have
\n $\int x_n$ and y_n have
\nthe same distribution.
\n $\int x_n = y_n$ and y_n have
\n $\int x_n = y_n$ and $\int x_n$ and y_n have
\n $\int x_n = y_n$ and $\int x_n$
\n $\int x_n = y_n$ and $\int x_{n-1} = x_n$ and $\int x_{n-2} = y_n$
\n $\int x_{n-1} = \sum_{n=1}^{n} \sum_{x} P(\tau = m, X_m = x) P(X_n = y | X_m = x)$
\n $= \sum_{n=1}^{n} \sum_{x} P(\tau = m, Y_m = x) P(X_n = y | X_m = x)$
\n $= \sum_{n=1}^{n} \sum_{x} P(\tau = m, Y_m = x) P(X_n = y | Y_n = x)$
\n $= \sum_{n=1}^{n} \sum_{x} P(\tau = m, Y_m = x) P(X_n = y_n - x_n)$
\n $= \sum_{n=1}^{n} \sum_{x} P(X_n = y_n)$
\n $= \sum_{n=1}^{n} \sum_{x} P(X_n = y_n - x_n)$
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\n $= \sum_{n=1}^{n} \sum_{x} P(X_n =$

Thus holds regardless of how we initially
\nlike
$$
X_0
$$
 and \overline{Y}_0 . Assume $X_0 = x$ and
\n $Y_0 \sim \pi$. Then, (0) gives
\n $W_0 \times \pi$. Then, (0) gives
\n $W_0 (X_0 = \cdot) - \pi(\cdot)W_{\tau} = \mathbb{P}(T > n) \rightarrow 0$
\n $W_0 (X_0 = \cdot) - \pi(\cdot)W_{\tau} = \mathbb{P}(\tau > n) \rightarrow 0$
\nfrom, the result follows from the Lemma
\n W_0 to get below from the Lemma
\n W_0 to get below τ