Lecture 18

Wed Mar/27/2024

Last time

- Recap
- Recurrence and transience

Today

- Stationary measures
- Existence
- Uniqueness

Stationary Measures

Let S be the state space of a Markov chain. Today we will cover stationary or invariant measures.

Def: Let X_n be a MC with countable state space S. A measure μ on S is **stationary** if

$$\underbrace{\sum_{x \in S} \mu(x)\, p(x,y)}_{\text{"Prob." of getting to } y \text{ after initializing with } \mu \text{ and running one step of the MC.}} = \mu(y)$$

If μ is a prob. measure $\Rightarrow$ μ is a stationary dist.

Lemma: If μ is stationary distribution,

$$\Rightarrow \quad \mathbb{P}_\mu(X_n = y) = \mu(y) \quad \forall n.$$

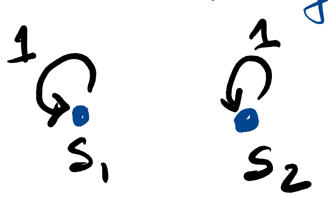
Proof: Exercise. $\square$

Example (Asymmetric random walk): Let $S_n = \sum_{k=1}^{n} \xi_k$ with $\mathbb{P}(\xi_k = 1) = p = 1 - \mathbb{P}(\xi_k = -1)$. Let $p > q := 1-p$. Then $\mu(x) = \left(\frac{p}{q}\right)^x$ is a stationary measure. Note that

$$\begin{aligned}
\sum_{x \in S} \left(\frac{p}{q}\right)^x p(x,y) &= \mu(y+1)\,p(y+1,y) + \mu(y-1)\,p(y-1,y) \\
&= (p/q)^{y+1} q + (p/q)^{y-1} p \\
&= (p/q)^{y} (p+q) \\
&= (p/q)^{y}.
\end{aligned}$$

However, μ is not a dist. since $\sum_x \mu(x) = \infty$. $\dashv$

Example (Lack of uniqueness): Consider



Both $\mu_1(x) = \mathbb{1}_{\{x = s_1\}}$ and $\mu_2(x) = \mathbb{1}_{\{x = s_2\}}$ are stationary distributions. $\dashv$

Example (Lack of existence): Consider

$$\bullet \xrightarrow[1]{} \bullet \xrightarrow[1]{} \bullet \xrightarrow[1]{} \bullet \longrightarrow \cdots$$

Assume μ is a nonzero stationary measure, then $\exists\, y$ s.t $\mu(y) > 0$,

$$\mu(y) = \sum \mu(x)\, p(x,y) = \mu(y-1)\, p(y-1,y) \overset{\text{induction}}{=} \overset{1}{\cancel{\prod_{x=0}^{y} p(x,x+1)}}\,\mu(0)$$

But also, $\mu(0) = \sum \mu(x) p(x,0) = 0$ $\lightning$ $\dashv$

Existence

Theorem Let X_n be a MC with countable state space. Assume that $\exists x$ recurrent. Let $T_x = \inf\{n \geq 1 : X_n = x\}$, then

$$\mu_x(y) = \mathbb{E}_x\left(\sum_{n=0}^{T_x} \mathbb{1}_{\{X_n = y\}}\right) = \sum_{n=0}^{\infty} \mathbb{P}(X_n = y, T_x > n)$$

defines a stationary measure. $\dashv$

Number of times we visit y in one cycle.

Remark: When all states are transcient, a stationary measure may or may not exist (see examples above).

Proof: Define $\bar{p}_n(x,y) = \mathbb{P}_x(X_n = y, T_x > n)$. By Fubini's

$$\sum_{y \in S} \mu_x(y) p(y,z) = \sum_{y \in S} \sum_{n=0}^{\infty} \bar{p}_n(x,y) p(y,z)$$

$$= \sum_{n=0}^{\infty} \sum_{y \in S} \bar{p}_n(x,y) p(y,z) = \circledast.$$

We want to show that $\circledast = \mu_x(z)$.

Consider two cases

Case 1: $z \neq x$

$$\sum_y p_n(x,y)p(y,z) = \sum \mathbb{P}_x(\underbrace{X_n = y, X_{n+1} = z, T_x > n}_{\bar{p}_{n+1}(x,z)})$$

since $T > n$ and we are summing every possibility.

$$= \mathbb{P}_x(X_{n+1} = z, T_x > n+1)$$

Then $\circledast = \sum_{n=0}^{\infty} \bar{p}_{n+1}(x,z) = \sum_{n=0}^{\infty} \bar{p}_n(x,z)$

$\bar{p}_0(x,z) = 0$ since $x \neq z$.

$$= \mu_x(z)$$

Case 2: $z = x$

$$\sum_y p_n(x,y)p(y,x) = \sum \mathbb{P}_x(X_n = y, T_x > n, X_{n+1} = x)$$

$$= \mathbb{P}(T_x = n+1).$$

Then, $\circledast = \sum_{n=0}^{\infty} \mathbb{P}(T_x = n+1) = \sum_{n=0}^{\infty} \mathbb{P}(T_x = n)$

$\mathbb{P}(T_x = 0) = 0.$

$$= 1 = \mu_x(x).$$

$\square$

Uniqueness

We saw an example where we had two different stationary measures, it is easy to see that their conic hull would also be stationary.

Exercise (♡): Assume that μ_1 and μ_2 are stationary $\Rightarrow$ for any $\alpha, \beta \in \mathbb{R}$ we have that $\nu = \alpha\mu_1 + \beta\mu_2$ is stationary provided that $\nu(x) \geq 0 \quad \forall x$ and $\exists x \quad \nu(x) > 0$ $\dashv$

Broadly speaking, uniqueness fails because of two reasons:

- Multiple irreducible classes (↺ ↺).
- Lack of recurrance (In the asymmetric walk example $\mu(x) = 1$ is stationary).

The following Theorem formalizes this claim.

Theorem: If a MC is irreducible and recurrent (all states are recurrent), then, there is a unique stationary measure up to constants.

Proof: Fix $x \in S$, let μ be a stationary measure, we shall prove that $\mu = \mu_x$ (up to constants). WLOG set $\mu(x) = 1$

Let $y \neq x$, by assumption

$$\mu(y) = \sum_{z_0 \in S} \mu(z_0) p(z_0, y)$$

$$= p(x,y) + \sum_{z_0 \neq x} \mu(z_0) p(z_0, y)$$

$\mu(x) = 1.$

$$= \mathbb{P}_x(X_1 = y, T_x > 1) + \underbrace{\sum_{z_0 \neq x} \mu(z_0) p(z_0, y)}_{T_1}.$$

Repeating the argoment

$$T_1 = \sum_{z_0 \neq x} \left(\sum_{z_1 \in S} \mu(z_1) p(z_1, z_0) \right) p(z_0, y)$$

$$= \sum_{z_0 \neq x} p(x, z_0) \quad p(z_0, y)$$

$$+ \sum_{z_0, z_1 \neq x} \mu(z_1) p(z_1, z_0) p(z_0, y)$$

$$= \mathbb{P}_x(X_2 = y, T > 2) +$$

$$\sum_{z_0, z_1 \neq x} \mu(z_1) p(z_1, z_0) p(z_0, y)$$

Inductively we obtain

$$\mu(y) \geq \sum_{m=1}^{n} \mathbb{P}[X_n = y, T_x > n] \uparrow \mu_x(y).$$

Searching contradiction assume that $\mu(y) - \mu_x(y) > 0$ for some y.

Then by Exercise (♡) we get that $\nu = \mu - \mu_x$ is stationary. Further since $\mu(x) = \mu_x(x) = 1$, $\nu(x) = 0$.

Let k be such that $p^{(k)}(y,x) > 0$.

Then,

$$0 = \nu(x) = \sum_{z \in S} \nu(z)\, p^{(k)}(z,x) = \nu(y)\, p^{(k)}(y,x) > 0.$$

↯ $\square$

So $\mu = \mu_x$.