Lecture 17 Mon Mar/25/2024

Announcements

- I will release a list of potential projects this week.
- You'll have a weak to pick 1 and 3 weeks to prepare a presentation ← 1 hour (last week of class) and 4 to prepare a document. (Groups of 3!)

Last time

- Markov Property continued
- Strong Markov Property
- Applications

Today

- Recap
- Recurrence and transience

Recap

From now on, we assume S is countable.

Recall last time we close with

Let $T_y^0 = 0$ and for $k \geq 1$, let

$$T_y^{(k)} = \inf\{n > T_y^{(k-1)} : X_n = y\}.$$

↑ kth time we visit y.

We let $T_y = T_y^{(1)}$ and $\rho_{xy} = \mathbb{P}(T_y < \infty)$

Theorem: Assume S is countable:

$$\mathbb{P}_x(T_y^k < \infty) = \rho_{xy}\,\rho_{yy}^{k-1} \qquad \dashv$$

Today we go back to understanding states that we go back to.

Def: A state x is **recurrent** if $\rho_{xx} = 1$.
A state x is **transcient** if $\rho_{xx} < 1$. $\dashv$

Example: In HW2 you prove that if $S_n = \sum^n \xi_k$ with $\mathbb{P}(\xi_k = 1) = \mathbb{P}(\xi_k = -1) = \frac{1}{2}$, then $x = 0$ is recurrent. When $\mathbb{P}(\xi_k = 1) > \frac{1}{2}$ $x = 0$ is transcient, can you prove it? $\dashv$

Let's explore a few properties of recurrent states.

Number of times we visit y.

Let $N(y) = \sum_{n=1}^{\infty} \mathbb{1}_{\{X_n = y\}}$.

Theorem: A state is recurrent if, and only if, $\mathbb{E}_x N(x) = \infty$.

Proof: The result follows from

$$\mathbb{E}_x N(y) = \sum_{k=1}^{\infty} \mathbb{P}_x(N(y) \geq k)$$

$$= \sum_{k=1}^{\infty} \mathbb{P}_x(T_y^k < \infty) \qquad (♡)$$

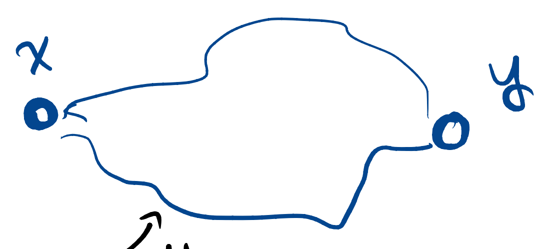
$$= \sum_{k=1}^{\infty} \rho_{xy}\, \rho_{yy}^{k-1}$$

$$= \frac{\rho_{xy}}{1-\rho_{yy}}. \qquad \square$$

The next result shows that recurrance is contagious.

Theorem: If x is recurrent and $\rho_{xy} > 0$ $\Rightarrow$ y is recurrent and $\rho_{yx} = 1$.

Intuition



If there is a path to from x to y, then we have a Bernoulli modeling if we go through y before going back to x.

Proof: First we prove that $\rho_{yx} = 1$. Seeking contradiction, assume $\rho_{yx} < 1$. Let

$$K = \inf\{k : p^{(k)}(x,y) > 0\}$$

Prob of getting from x to y in k steps.

There is a path $x \to y_1 \to \cdots \to y_{K-1} \to y$ so that

$$p(x,y_1)\, p(y_1,y_2) \cdots p(y_{K-1},y) > 0.$$

Since K is minimal $y_i \neq x$ $\forall i$, then

$$\mathbb{P}_x(T_x = \infty) \geq p(x,y_1)\, p(y_1,y_2) \cdots p(y_{K-1},y)\,(1-\rho_{yx})$$

$$> 0$$

↯

Thus, $\rho_{yx} = 1$.

Now we prove that y is recurrent.

Since $\rho^{yx} = 1$, we have $\exists L$ s.t. $p^{(L)}(y,x) > 0$. Note that

$$p^{(L+n+K)}(y,y) \geq p^{(L)}(y,x)\, p^{(n)}(x,x)\, p^{(K)}(x,y).$$

Then,

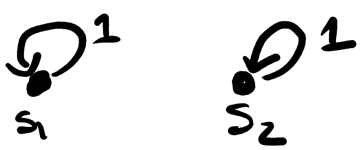
$$\begin{aligned}
\mathbb{E}_y N(y) &= \sum_{n=1}^{\infty} \mathbb{E}_y \mathbb{1}_{\{X_n = y\}} \\
&= \sum_{n=1}^{\infty} p^{(n)}(y,y) \\
&\geq \sum_{n=1}^{\infty} p^{(L+n+K)}(y,y) \\
&\geq p^{(L)}(y,x)\, p^{(K}(y,x) \sum_{n=1}^{\infty} p^{(n)}(x,x) \\
&> \infty.
\end{aligned}$$

$\square$

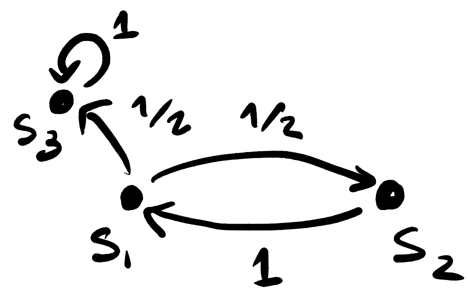
Now we introduce to important concepts that will help us decompose chains:

Def: A set $C \subseteq S$ is **closed** if $\forall x \in C$ and $y \in S$ if $\rho_{xy} > 0 \Rightarrow y \in C$. A set $D \subseteq S$ is irreducible if $\forall x, y \in D$ we have $\rho_{xy} > 0$.

Example:



$\{s_1, s_2\}$ is closed
not irreducible



$\{s_1, s_2\}$ is irreducible
not closed.

Theorem: Let $C \subseteq S$ be a finite closed state. Then, C contains a recurrent state. If C is irreducible then all states in C are recurrent.

Intuition: If a set is closed and irreducible we will stay in C and go back to each of its states i.o.

Proof: The second claim follows follows from Theorem ☒. We focus on the first claim. Suppose seeking contradiction that $\forall y \in C$ $\rho_{yy} < 1$, but this implies

$$\infty > \sum_{y \in C} \mathbb{E}_x N(y) = \sum_{y \in C} \sum_{n=1}^{\infty} p^{(n)}(x,y) = \sum_{n=1}^{\infty} \sum_{y \in C} p^{(n)}(x,y) = \sum_{n=1}^{\infty} 1.$$

Since $|C| < \infty$ and (☒).

C is closed

↯ ☐

Theorem: Let $R = \{x : \rho_{xx} = 1\}$ be recurrent states of a Markov Chain. Then, $R = \bigcup_i R_i$ where the R_i are irreducible, closed, disjoint sets.

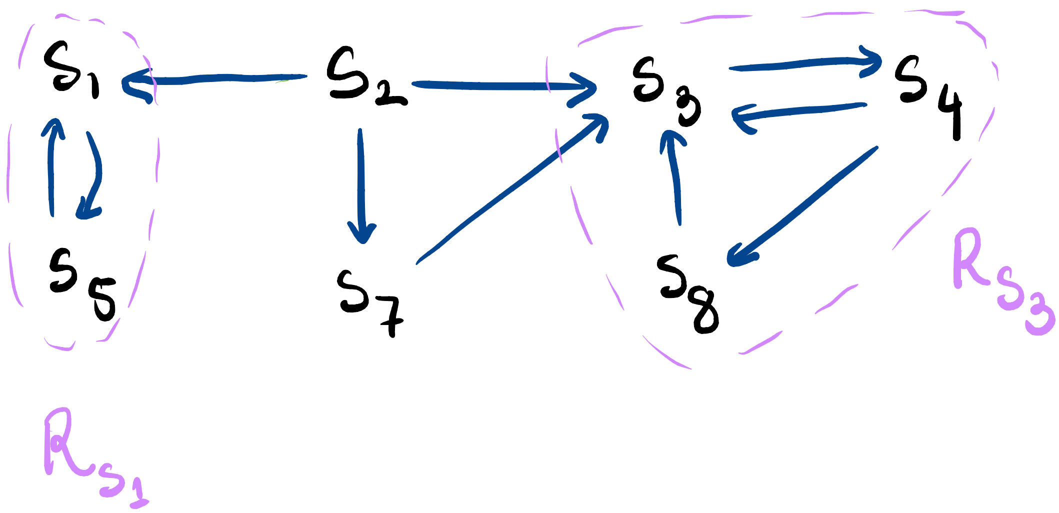
Proof: For any recurrent state x, define $R_x = \{y : \rho_{xy} > 0\}$. Consider the collection $\{R_x\}$, we claim that this gives the desired partition. For a given x, by Theorem ☒, if $y \in C_y \Rightarrow \rho_{yx} = 1 > 0$.

Closed: Let $w \in C_x$ and z s.t. $\rho_{wz} > 0$ $\Rightarrow \rho_{xz} \geq \rho_{xw} \rho_{wz} > 0 \Rightarrow z \in C_x$.

Irreducible: Let $w, z \in C_x \Rightarrow \rho_{wz} \geq \underbrace{\rho_{wx}}_{=1} \rho_{xw} > 0$

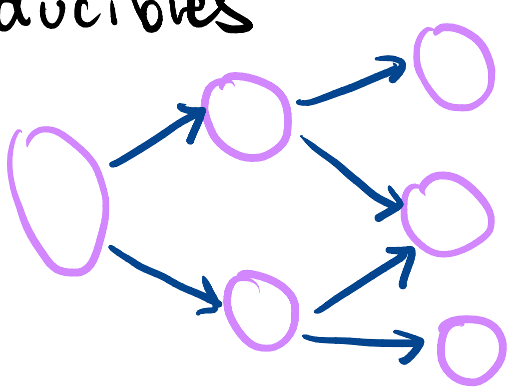
Disjoint: Assume $z \in C_x \cap C_y$. Since C_x is irreducible z is connected to all w in C_x $\Rightarrow$ $y \to z \to w$, thus $C_x \subseteq C_y$ and similarly $C_y \subseteq C_x$. Thus either $C_x = C_y$ or $C_x \cap C_y$. $\square$

Example:



More generally we can decompose into irreducible components $R_x = \{y : p_{xy} > 0 \text{ and } p_{yx} > 0\}$, which gives a DAG of irreducibles

Can you prove it?



Where the leaves of the DAG are closed irreducible comp.