Lecture 16

Last time

- Formal construction continued.
- Markov Property

Today

- Strong Markov Property
- Applications

Lemma 2: For bounded measurable f_m, we have

$$\mathbb{E}_\mu\left[\prod_{m=0}^{n} f_m(X_m)\right] = \int \mu(dx_0)\, f_0(x_0) \int p(x_0, dx_1)\, f_1(x_1) \cdots \int p(x_{n-1}, dx_n)\, f_n(x_n).$$

Proof of Lemma 2: By Tower Law

$$\mathbb{E}_\mu\left[\prod_{m=0}^{n} f_m(X_m)\right] = \mathbb{E}_\mu\, \mathbb{E}\left[\prod f_m(X_m) \mid \mathcal{F}_{n-1}\right]$$

Take out what you know $= \mathbb{E}_\mu\left[\prod^{n-1} f_m(X_m)\, \mathbb{E}\left[f_n(X_n) \mid \mathcal{F}_{n-1}\right]\right]$

Lemma 1 $= \mathbb{E}_\mu\left[\prod^{n-1} f_m(X_m) \int p(X_{n-1}, dx_n)\, f_n(x_n)\right]$

The result follows by recusing this argument $\square$

Proof of Markov 1: We prove for a simpler case and extend using $\pi-\lambda$ Thm.

Consider $Y(\omega) = \prod_{k=1}^{n} g_k(\omega_k)$ with g_k bounded and measurable. Let $A = \{\omega : \omega_0 \in A_0, \ldots, \omega_m \in A_m\}$

Let $f_k = \begin{cases} \mathbb{1}_{A_k} & k < m, \\ g_0 \mathbb{1}_{A_m} & k = m, \\ g_{k-m} & k > m. \end{cases}$ and apply Lemma 2:

$$E_\mu[\mathbb{1}_A Y \circ \theta_m] = E_\mu[\mathbb{1}_A \prod_{k=0}^{n} g_k(X_{m+k})]$$

$$= \int_{A_0} \mu(dx_0) \int_{A_1} p(x_0, dx_1) \cdots \int_{A_m} p(x_{m-1}, dx_m)$$

$$\underbrace{\Big[g_0(x_m) \int p(x_m, dx_{m+1}) g_1(x_{m+1}) \cdots \int p(x_{m+n-1}, dx_{m+n}) g_n(x_{m+n}) \Big]}_{E_{x_m} Y} \quad (\star)$$

$$= E_\mu[\mathbb{1}_A E_{X_m} Y].$$

Again using the π-λ $(\star)$ extends to any $A \in \mathcal{F}_m$. So the result holds for $Y = \prod^n g_k(\omega_k)$. To finish the proof we use the Mono. Class Thm: Let $\mathcal{H}$ be the collection of Y for which $(\heartsuit)$ holds, let $\mathcal{A} = \{\{\omega : \omega_0 \in A_0, \dots, \omega_k \in A_k\} : \forall k\ \forall A_i \in \mathcal{S}\}$, taking $g_m = \mathbb{1}_{A_m}$ in $(\star)$ shows that (i) holds, and $\mathcal{H}$ (ii) and (iii) by linearity $\Rightarrow$ $(\heartsuit)$ holds for bounded measurable functions. $\square$

Strong Markov Property

We prove an extension of Markov 1.

Let N be a stopping time. Define

$$\mathcal{F}_N = \{ A : A \cap \{N=n\} \in \mathcal{F}_n \ \forall n \}.$$

Define the random shift operator

$$\theta_N(\omega) := \begin{cases} \theta_n(\omega) & \text{on } \{N=n\} \\ * & \text{on } \{N=\infty\} \end{cases}$$

This is a formal symbol we add to Ω.
No need to worry about it since we assume $N<\infty$

Theorem (Markov 2): For each n, let $Y_n : \Omega_0 \to \mathbb{R}$ be measurable with $|Y_n| \le M$ for some $M>0$. Then,

$$\mathbb{E}_\mu [Y_N \circ \theta_N \mid \mathcal{F}_N] = E_{X_N} Y_N \text{ on } \{N<\infty\}.$$

This is $\varphi(x,n) = E_x(Y_N)$ evaluated at $x = X_N$, $n = N$.

Proof: Let $A \in \mathcal{F}_N$. Let's partition depending on the value of N,

$$\mathbb{E}_\mu [(Y_N \circ \theta_N) \mathbb{1}_{A \cap \{N<\infty\}}]$$

$$\circledast = \sum_{n=0}^{\infty} \mathbb{E}_\mu[(Y_N \circ \theta_N) \mathbb{1}_{A \cap \{N=n\}}]$$

Notice that $A \cap \{N=n\} \in \mathcal{F}_n$ so

Markov 1 gives

$$\circledast = \sum_{n=0}^{\infty} \mathbb{E}_\mu[(\mathbb{E}_{X_n} Y_n]) \mathbb{1}_{A \cap \{N=n\}}]$$

$$= \mathbb{E}_\mu[(\mathbb{E}_{X_N} Y_N) \mathbb{1}_{A \cap \{N<\infty\}}]. \qquad \square$$

Applications

We will use Markov 2 to prove a formula that will be key in the next few lectures.

Assume that S is countable.

Let $T_y^0 = 0$ and for $k \geq 1$, let

$$T_y^{(k)} = \inf\{n > T_y^{(k-1)} : X_n = y\}.$$

↑ kth time we visit y.

We let $T_y = T_y^{(1)}$ and $\rho_{xy} = \mathbb{P}(T_y < \infty)$.

Theorem: Assume S is countable:

$$\mathbb{P}_x(T_y^k < \infty) = \rho_{xy} \rho_{yy}^{k-1}$$

Proof If $k=1$, it follows by definition.
Suppose $k \geq 2$, define

$$Y(\omega) = \mathbb{1}_{\{\exists X_n = y\}}(\omega) = \begin{cases} 1 & \exists \omega_n = y, \\ 0 & \text{otherwise.} \end{cases}$$

Set $N = T_y^{(k-1)} \Rightarrow Y \circ \theta_N = 1$ if $T_y^{(k)} < \infty$.
Markov 2 states that on $\{N < \infty\}$

$$\mathbb{E}_x [Y \circ \theta_N \mid \mathcal{F}_N] = \mathbb{E}_{X_N} Y$$

$X_N = y$ on $\{N<\infty\}$ $\rightarrow$ $$= \mathbb{E}_y Y$$

(♡) $$= \mathbb{P}_y (T_y < \infty)$$

$$= \rho_{yy}.$$

We can now analyze the probability we care about

$$\mathbb{P}_x (T_y^k < \infty) = \mathbb{E}_x [(Y \circ \theta_N) \mathbb{1}_{\{N<\infty\}}]$$

$$= \mathbb{E}_x [\mathbb{E}_x [Y \circ \theta_N \mid \mathcal{F}_N] \mathbb{1}_{\{N<\infty\}}]$$

(♡) $$= \mathbb{E}_x [\rho_{yy} \mathbb{1}_{\{N<\infty\}}]$$

$$= \rho_{yy} \mathbb{P}_x[T_y^{(k-1)} < \infty]$$

Induction $\rightarrow$ $\vdots$

$$= \rho_{yy}^{k-1} \mathbb{P}_x[T_y < \infty]$$

$$= \rho_{yy}^{k-1} \rho_{xy}.$$

$\square$

A second application:

Theorem: Let $\xi_1, \xi_2, \ldots$ be iid rv with a symmetric distribution around 0. Then

$$\mathbb{P}\left(\sup_{m \leq n} S_m \geq a\right) \leq 2\mathbb{P}(S_n \geq a)$$

$\dashv$

Exercise for the break read the proof in Durrett (Thm 5.2.7) and interiorize the ideas.