Lecture 15 Mon Mar/11/2024

Last time

- Intro to Markov Chains
- Formal Construction

Today

- Formal construction
- Markov Property.

Formal construction continued

Recall that given a measure for X_0 μ we defined a measure over $(S^n, \mathcal{S}^n)$
$\mathbb{P}(X_j \in B_j, 0 \le j \le n) =$

$$\int_{B_0} \mu(dx_0) \int_{B_1} p(x_0, dx_1) \cdots \int_{B_n} p(x_{n-1}, dx_n).$$

which we then extended via Kolmogorov's extension Theorem to a measure $\mathbb{P}_\mu$ on $(\Omega_0, \mathcal{F}_\infty) = (S^{\mathbb{N}}, \mathcal{S}^{\mathbb{N}})$

Note that this construction yields measures for each $x \in S$ via $\mu = \delta_x$, we use $\mathbb{P}_x = \mathbb{P}_{\delta_x}$. Further, for an arbitrary μ

$$\mathbb{P}_\mu(A) = \int \mu(dx)\, \mathbb{P}_x(A).$$

Let $\mathbb{E}_x$ as well.

Our goal today is to prove several

versions of Markov's Property. Recall $X_n(\omega) = \omega_n$.

Theorem (Markov 0) X_n is a Markov chain with respect to $\mathcal{F}_n = \sigma(X_0, X_1, \dots, X_n)$ with transition transition probability p, i.e.,

$$\mathbb{P}_\mu(X_{n+1} \in B \mid \mathcal{F}_n) = p(X_n, B).$$

Proof: We show that $p(X_n, B)$ is a version of $\mathbb{E}[\mathbb{1}_{\{X_{n+1} \in B\}} \mid \mathcal{F}_n]$. $p(X_n, B)$ is clearly $\mathcal{F}_n$ measurable. Let $A = \{X_0 \in B_0, X_1 \in B_1, \dots, X_n \in B_n\}$, $B_{n+1} = B$. By definition

$$\int_A \mathbb{1}_{\{X_{n+1} \in B\}} \, d\mathbb{P}_\mu = \mathbb{P}_\mu(A \cap \{X_{n+1} \in B\})$$

$$= \int_{B_0} \mu(dx_0) \int_{B_1} p(x_0, dx_1) \cdots \int_{B_n} p(x_{n-1}, dx_n)\, p(x_n, B)$$

We would like to say that

$$\stackrel{?}{=} \int_A p(X_{n+1}, B) \, d\mathbb{P}_\mu. \qquad (?)$$

To do so we follow a standard pipeline, note that for any $C \in \mathcal{S}$

$$\int_A \mathbb{1}_C(X_n) \, d\mathbb{P}_\mu = \int_{B_0} \mu(dx_0) \cdots \int_{B_n} p(x_{n-1}, dx_n)\, \mathbb{1}_C(x_n).$$

Then, we have equality for simple functions and by BCT the equality is valid for bounded measurable functions and so (?) follows.

A simple computation reveals that the set A s.t.

$$\int_A \mathbb{1}_{\{X_{n+1} \in B\}} \, d\mathbb{P}_\mu = \int_A p(X_{n+1}, B) \, d\mathbb{P}_\mu \quad (\star)$$

forms a λ-system. Moreover, we proved that this equality holds for $A = \{X_0 \in B_0, \dots, X_n \in B_n\}$ which forms a π-system. By the π-λ Theorem (Thm 2.1.6 in Durrett) equality $(\star)$ holds $\forall A \in \mathcal{F}_n$, which proves the result. $\square$

Next we prove a couple of extensions of the Markov Property where $\mathbb{1}_{\{X_{n+1} \in B\}}$ is substituted by a bounded fun. of the future, $h(X_n, X_{n+1}, \dots)$. Let $\theta_m : \Omega_0 \to \Omega_0$ given by

$$\theta_m(\omega_0, \omega_1, \dots) = (\omega_m, \omega_{m+1}, \dots).$$

Theorem (Markov 1) Let $Y : \Omega_0 \to \mathbb{R}$ be bounded and measurable. Then

(b) $\mathbb{E}_\mu(Y \circ \theta_m \mid \mathcal{F}_m) = \mathbb{E}_{X_m} Y$ ← Expectation of $Y(X_m, X_{m+1}, \dots)$ with X_m fixed.

$\dashv$

Corollary (Chapman-Kolmogorov) If S is countable, then

$$\mathbb{P}_x(X_{n+m} = z) = \sum_{y \in S} \mathbb{P}_x(X_m = y)\, \mathbb{P}_y(X_n = z).$$

Proof:

$$\begin{aligned}
\mathbb{P}_x(X_{n+m} = z) &= \mathbb{E}_x[\mathbb{1}_{\{X_{n+m}=z\}}] \\
&= \mathbb{E}_x[\mathbb{E}[\mathbb{1}_{\{X_{n+m}=z\}} \mid \mathcal{F}_m]] \\
&= \mathbb{E}_x[\mathbb{E}_{X_m}[\mathbb{1}_{\{X_n=z\}}]] \\
&= \mathbb{E}_x[\mathbb{P}_{X_m}(X_n = z)] \\
&= \sum_{y \in S} \mathbb{P}_x(X_m = y)\, \mathbb{P}_y(X_n = z). \qquad \square
\end{aligned}$$

Markov 1 → $\mathbb{1}_{\{X_{n+m}=z\}} = \mathbb{1}_{\{X_n=z\}} \circ \theta_m$

To prove Markov 1 we leverage

Theorem (Monotone Class Theorem) Let $\mathcal{A}$ be a π-system containing Ω and $\mathcal{H}$ be a collection of functions that satisfies:

(i) If $A \in \mathcal{A} \Rightarrow \mathbb{1}_A \in \mathcal{H}$.

(ii) If $f, g \in \mathcal{H} \Rightarrow f+g \in \mathcal{H}$ and $cf \in \mathcal{H}$ $\forall c \in \mathbb{R}$.

(iii) If $f_n \in \mathcal{H}$ are nonnegative and increase to a bounded function $f \Rightarrow f \in \mathcal{H}$.

Then, $\mathcal{H}$ contains all bounded functions measurable with respect to $\sigma(\mathcal{A})$. $\dashv$

Exercise prove this using the π-λ Thm.

Lemma 1: For any bounded measurable f:

$$\mathbb{E}[f(X_{n+1}) \mid \mathcal{F}_n] = \int p(X_n, dy)\, f(y).$$

Proof of Lemma 1: Let $\mathcal{H}$ be the collection of functions for bounded functions so that the identity holds. Let $\mathcal{A} = \{ \mathbb{1}\{X_{n+1} \in B\} \}$

Markov show that (i) holds so the result follows from the Mono. Class Thm. $\dashv$

TO BE CONTINUED...