lecture 14 Wed Mar/6/2024

Last time

- Some examples
- summary

Today

- Intro to Markov Chains
- Formal Construction

Intro to Markov Chains

A Markov Chain is a random process $(X_n)_n$ taking values in some state space S. Let's start with the simpler case where S is countable. In which case, the "Markov property" reads: $\forall\, i_0, i_1, \dots, i_{n-1}, i, j \in S$ we have

$$P(X_{n+1}=j \mid X_n=i, X_{n-1}=i_{n-1}, \dots, X_0=i_0)$$
$$= P(X_{n+1}=j \mid X_n=i).$$

We call this the transition probability

$$p(i,j) = P(X_{n+1}=j \mid X_n=i).$$

Q: Why should we care?

Many random process are Markov

Chains, and this simple property leads to a rich and useful theory.

Examples

- **Random walks**

Let $\xi_1, \xi_2, \ldots \in \mathbb{R}^d$ be iid with distribution μ and let

$X_n = X_0 + \xi_1 + \ldots + \xi_n$ for constant X_0.

Then, X_n is a Markov Chain (why?) with transition probability

$$p(i,j) = \mu(\{j-i\}).$$

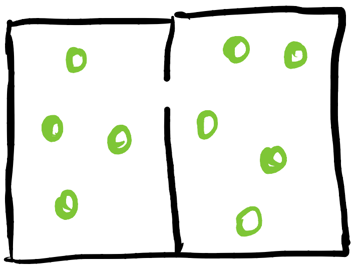
- **Branching Process**

Let $S = \{0, 1, 2, \ldots\}$ and $\xi_1, \xi_2, \ldots$ be iid nonnegative integer-valued r.v.. Then the branching process we cover in Lecture 9 defines a Markov Chain via

$$p(i,j) = \mathbb{P}\left(\sum_{k=1}^{i} \xi_k = j\right).$$

▷ Ehrenfest Chain

Assume we have r particles in two chambers connected by a small hole.



At any moment in time a random particle jumps from one chamber to the other. This process yields a Markov Chain with $S = \{0, \dots, r\}$ and

$$p(i,j) = \begin{cases} (r-i)/r & \text{if } j = i+1 \\ i/r & \text{if } j = i-1 \\ 0 & \text{otherwise.} \end{cases}$$

$r \times r$ matrix.

What happens in the long run?

▷ Wright - Fisher model

Assume we have a constant-size population (say size N), and there are two allele types A and a. Assume that at each generation we draw

N new individuals by sampling with replacement and we will like to understand the dynamics of

X_n = "Number of A alleles at generation n."

This is a Markov Chain with $S = \{0, 1, \ldots, N\}$ and

$$p(i,j) = \binom{N}{j}\left(\frac{i}{N}\right)^j\left(1-\frac{i}{N}\right)^{N-j}$$

Note that the state 0 and N are absorbing, i.e., $p(i,i) = 1$.

Formal construction

We want to talk about conditio-
nal probabilities and we saw
that in full generality they can
be tricky. So we will restrict
ourselves to nice spaces.

Def: We say that a measurable
space $(S, \mathcal{S})$ is **nice** if there a 1-1
map $\varphi : S \to \mathbb{R}$ so that φ and φ^{-1}
are measurable. $\dashv$

Fact (Theorem 2.1.22) If S is a Bo-
rel subset of a complete separable
metric space, and $\mathcal{S}$ is the collection
of Borel subsets of S, then $(S, \mathcal{S})$
is nice. $\dashv$

I.e, most spaces we encounter are
nice.

Fact (Theorem 4.1.17) If $(S, \mathcal{S})$ is
nice then regular conditional probabi-

lities exist.

I.e., we can take $\mathbb{P}(A|\mathcal{F}) = \mathbb{E}(\mathbb{1}_A|\mathcal{F})$.

With this it makes sense to define

Def: A function $p : S \times \mathcal{S} \to [0,1]$ is said to be a *transition probability* if:

i) For each $\omega \in S$, $A \mapsto p(\omega, A)$ is a probability measure on $(S, \mathcal{S})$.

ii) For each $A \in \mathcal{S}$, $\omega \mapsto p(\omega, A)$ is a measurable function.

We say that X_n is a Markov Chain with respect to a filtration $\mathcal{F}_n$ if

$$\mathbb{P}(X_{n+1} \in B \mid \mathcal{F}_n) = p(X_n, B).$$

Given a distribution μ for X_0, we can define

$$\mathbb{P}(X_j \in B_j, 0 \le j \le n) =$$

$$\int_{B_0} \mu(dx_0) \int_{B_1} p(x_0, dx_1) \cdots \int_{B_n} p(x_{n-1}, dx_n).$$

This defines a probability distribution

for a finite tuple $(X_0, X_1, \ldots, X_n)$. Can we extend it to a distribution of $(X_0, \ldots)$ that matches the marginals of any finite tuple?

When $(S, \mathcal{S})$ is nice, this is exactly what Kolmogorov's extension Theorem (Thm 2.1.21 Durrett) get us. We can define a probability measure $\mathbb{P}_\mu$ on the sequence space

$$(\Omega_0, \mathcal{F}_\infty) = (S^{\mathbb{N}}, \mathcal{S}^{\mathbb{N}}).$$

TO BE CONTINUED...