Lecture 13 Mon Mar/04/2024

Last time

- Optional stopping via UI
- Backwards martingales

Today

- Three examples
- Summary.

The setting

Suppose we consider the setting with $(\mathbb{R}, \mathcal{B})$ and set

$$\Omega = \{(\omega_1, \omega_2, \dots) \mid \omega_i \in \mathbb{R}\}$$
$$\mathcal{F} = \mathcal{B} \times \mathcal{B} \times \dots$$
$$X_n(\omega) = \omega_n \quad \forall n.$$

Let $\mathcal{E}_n$ be the sub-σ-algebra generated by events that are invariant under permutations that fix the tail $n+1, n+2, \dots$ and let $\mathcal{E} = \bigcap_n \mathcal{E}_n$ be the exchageable σ-algebra.

Example 1: Ballot Theorem

Consider two candidates B and T

assume that B receives β votes and T receives τ with $\beta > \tau$. What is the probability that B leads T throughout the counting? The answer is $\dfrac{\beta - \tau}{\beta + \tau}$.

To prove this consider iid r.v.s $\xi_1, \xi_2, \dots, \xi_n$ $(n = \beta + \tau)$ s.t.

$$\xi_i = \begin{cases} 0 & \text{if the ith vote goes to } B \\ 2 & \text{otherwise.} \end{cases}$$

assume $\mathbb{P}(\xi_i = 0) = \mathbb{P}(\xi_i = 2) = 1/2$.

Let $S_n = \sum_{i=1}^{n} \xi_i$, $X_{-j} = S_j / j$, and

$$\mathcal{F}_{-j} = \sigma(S_j, \dots, S_n).$$

Notice that

$$G = \{B \text{ leads } T \text{ throughout counting}\}$$

$$= \{ S_j < j \text{ for } 1 \le j \le n \}.$$

$2\tau_j < \beta_j + \tau_j$

$\Leftrightarrow 0 < \beta_j - \tau_j$

and so what we want to prove is equiv. to

$$\mathbb{P}(G \mid S_n) \overset{(\downarrow)}{=} \left(1 - \frac{S_n}{n}\right)^+ = \left(1 - \frac{2\tau}{\beta + \tau}\right)^+ = \left(\frac{\beta - \tau}{\beta + \tau}\right)^+$$

Let us show $(\downarrow)$. Note that if $S_n \geq n$ the result is trivially true. Assume $S_n < n$. Let us show that that X_{-j} is a backwards martingale.

Because of symmetry

$$\mathbb{E}[\xi_{j+1} \mid \mathcal{F}_{-(j+1)}] = \frac{1}{j+1} \sum_{k=1}^{j+1} \mathbb{E}[\xi_k \mid \mathcal{F}_{-(j+1)}]$$

$$= \frac{1}{j+1} \mathbb{E}[S_{j+1} \mid \mathcal{F}_{-(j+1)}]$$

$$= \frac{S_{j+1}}{j+1}.$$

Since $X_{-j} = (S_{j+1} - \xi_{j+1})/j$ we have that

$$\mathbb{E}[X_{-j} \mid \mathcal{F}_{-(j+1)}] = \frac{1}{j}\Big[\mathbb{E}[S_{j+1} \mid \mathcal{F}_{-(j+1)}] - \mathbb{E}[\xi_{j+1} \mid \mathcal{F}_{-(j+1)}]\Big]$$

$$(\star) \qquad = \frac{1}{j}\left[S_{j+1} - \frac{S_{j+1}}{j+1}\right]$$

$$= \frac{S_{j+1}}{j+1} = X_{-(j+1)}.$$

Let $N = \inf\{k \mid k \in \{-1, \dots, -n\},\ X_k \geq 1\}$

and set $N = -1$ if the set is empty.

N is the first point where T leads.

Note that on the event G^c $\exists\, N+1$ is such that $S_{N+1} < N+1 \Rightarrow S_N \le S_{N+1} \le N$

$\Rightarrow \frac{S_N}{N} \le 1$ and by def $X_N = 1$.

On the other hand, on the event G we have $N = -1$, then

$$X_N = X_{-1} = \frac{S_1}{1} < 1 \Rightarrow X_N = 0.$$

Thus, we have $X_N = \mathbb{1}_{G^c}$. Therefor

$$\mathbb{P}(G^c \mid X_{-n}) = \mathbb{E}[X_N \mid X_{-n}]$$

Follows from optional stopping why?

$$= X_{-n}$$

$$= \frac{S_n}{n}$$

$\dashv$

Example 2: Strong law of large numbers

Let $\xi_1, \xi_2, \dots$ iid r.v's with $\mathbb{E}|\xi_i| < \infty$.

Let $S_n = \sum_{i=1}^{n} \xi_i$, $X_{-n} = S_n/n$, and

$$\mathcal{F}_{-n} = \sigma(S_n, S_{n+1}, \dots).$$

Our goal is to show that

$$\frac{S_n}{n} \to \mathbb{E}\,\xi_1 \quad \text{a.s.}$$

The same computation as in (☆) we obtain that X_{-n} is a backwards martingales.

Fact (Hewitt-Savage) If $X_1, X_2, \dots$ are iid $\Rightarrow$ $\forall A \in \mathcal{E}$ $\mathbb{P}(A) \in \{0, 1\}$.

By the Convergence Thm for backwards martingales

$$\lim \frac{S_n}{n} \to \mathbb{E}[X_{-1} \mid \mathcal{F}_{-\infty}]$$

Since $\mathcal{F}_{-n} \subset \mathcal{E}_n \Rightarrow \mathcal{F}_{-\infty} \subseteq \mathcal{E}$ and by Hewitt-Savage the sets in $\mathcal{F}_{-\infty}$ are trivial, so we have $\mathbb{E}[X_{-1} \mid \mathcal{F}_{-\infty}] = \mathbb{E}[X_{-1}] = \mathbb{E}\xi_1$.

$\dashv$

Remark: One can use backwards martingales to prove Hewitt-Savage (see Example 4.7.6. in Durrett).

Example 8: de Finetti's Theorem

A sequence $X_1, X_2, \dots$ is said to be exchangeable if for every permutation π of $\{1, \dots, n\}$ we have

$$(X_1, \dots, X_n) \stackrel{d}{=} (X_{\pi(1)}, \dots, X_{\pi(n)}).$$

Equality in distribution

This generalize iid sequences, but these are more general as we can have a const. sequence $(X_1, X_1, X_1, \dots)$.

Theorem: If $X_1, X_2, \dots$ are exchangeable. Then, $\mathbb{E}[X_1 \mid \mathcal{E}], \mathbb{E}[X_2 \mid \mathcal{E}], \dots$ are iid.

This result is one of the pillars of Bayesian Statistics. This will be one of the potential topics for the final projects.

$\dashv$

Summary

In the last lectures we covered

- Conditional Expectation
- Martingales
- Stopping times
- Optional stopping
- Almost sure convergence.
- $\mathcal{L}^p$ convergence
- Uniform Integrability & $\mathcal{L}^1$ convergence.
- Backwards martingales.

Next we will tackle Markov Chains.