Probability Theory 2, Spring 2024 - Homework 5
Due before midnight on 5/9 (Gradescope)

*Your submitted solutions to assignments should be your own work. While discussing homework problems with peers is permitted, the final work and implementation of any discussed ideas must be executed solely by you. Acknowledge any source you consult.

Problem 1 - A step in the proof we were missing

Let $(\mathbb{R}, \mathcal{B})$ be the reals with the Borel sigma algebra. Let f_q be the density of $N(0, q)$, i.e., zero-mean normal with variance q. Recall the construction of the Brownian motion from Lecture 20, that for any set of times $0 \le t_1 < t_2 < \cdots < t_n \le 1$ we defined the distribution $\mu_{t_1,\ldots,t_n}$ on the product space $(\mathbb{R}^n, \mathcal{B}^n)$ via

$$\mu_{t_1,\ldots,t_n}(A_1, \ldots, A_n) = \int_{A_1} dx_1 \int_{A_2} dx_2 \cdots \int_{A_n} dx_n \prod_{m=1}^{n} f_{t_m - t_{m-1}}(x_m - x_{m-1}).$$

where $A_1, \ldots, A_n \in \mathcal{B}$. Show that the following compatibility condition holds: for any $0 \le t_1 < t_2 < \cdots < t_n \le 1$, $A_1, \ldots, A_n \in \mathcal{B}$, and $k \in \{1, \ldots, n\}$ we have

$$\mu_{t_1,\ldots,t_{k-1},t_{k+1},\ldots t_n}(A_1, \ldots, A_{k-1}, A_{k+1}, \ldots, A_n) = \mu_{t_1,\ldots t_n}(A_1, \ldots, A_{k-1}, \mathbb{R}, A_{k+1}, \ldots, A_n).$$

This is exactly what we needed to apply Kolmogorov's Extension Theorem.

Problem 2 - Let's stop one last time

Consider a right-continuous filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ (recall Lecture 22). Let T, S be stopping times. Show that the following are all stopping times:

(a) $\min\{S, T\}$,

(b) $\max\{S, T\}$,

(c) $S + T$.

Also, if T_n is a sequence of stopping times then so are

(d) $\liminf T_n$,

(e) $\limsup T_n$.

Problem 3 - Okay, now this is truly the last

Define $T_0 = \inf\{s > 0 \colon B_s = 0\}$ and let $L = \sup t \le 1 \colon B_t = 0$ (L here stands for last). Use the Markov Property at time $0 < t < 1$ to derive that

$$\mathbb{P}_0(L \le t) = \int f_t(y - 0)\mathbb{P}_y(T_0 > 1 - t)dy,$$

here f_t is defined as in Problem 1 and $\mathbb{P}_y$ is the probability measure given by initializing the Brownian motion at y.

Problem 4 - Brownian motion simulation

Subdivide the interval $[0, 1]$ into a regular grid of points $t_0 = 0, \ldots, t_n = 1$ for some fixed n. Write a script to generate Brownian motion starting at zero in the following manner:

Step 0 Set $B_0 = 0$.

Step 1 Generate n i.i.d. standard Gaussians $Z_1, \ldots, Z_n$.

Step 2 Recursively define $B_k = \sqrt{t_k - t_{k-1}} Z_k + B_{k-1}$ for all $k \in \{1, \ldots, n\}$.

Use this script with a sufficiently large n, e.g., $n = 5000$, to simulate one trajectory of the Brownian motion. Generate $N = 1000$ trajectories and plot them all together (maybe use different colors) and plot the curve $f(t) = \sqrt{t}$. How does it compare?