Probability Theory 2, Spring 2024 - Homework 4
Due one hour before lecture on 4/15 (Gradescope)

Your submitted solutions to assignments should be your own work. While discussing homework problems with peers is permitted, the final work and implementation of any discussed ideas must be executed solely by you. Acknowledge any source you consult.

Problem 1 - Things we didn't prove in class

Let X_n be a Markov Chain over a countable state space S. Establish the following facts.

(a) Show that if μ is a stationary distribution, then for all n we have $\mathbb{P}_\mu(X_n = y) = \mu(y)$.

(b) Assume that μ_1 and μ_2 are stationary measures and $\alpha_1, \alpha_2 \in \mathbb{R}$ are scalars such that $\nu(x) := \alpha_1\mu_1(x) + \alpha_2\mu_2(x) \geq 0$ for all $x \in S$. Show that ν is a stationary measure.

(c) Let $x, y \in S$ be recurrent states such that $\min\{\rho_{xy}, \rho_{yx}\} > 0$, then their periods coincide $d_x = d_y$.

Problem 2 - Positive recurrence and stationary distributions

In class, we showed that, under certain conditions, a stationary measure exists; in this question, we will understand when stationary distributions exist. Let X_n be an irreducible Markov chain with a countable state space. Recall that $T_x = \inf\{n \geq 1 \mid X_n = x\}$. We say that a state is positive recurrent if $\mathbb{E}_x[T_x] < \infty$ (which is stronger than recurrence). We will show that the following three are equivalent: (1) all states are positive recurrent, (2) at least one state is positive recurrent, and (3) there exists a stationary distribution.

(a) Recall the definition of $\mu_x(y) = \mathbb{E}_x\left[\sum_{n=1}^{T_x} \mathbf{1}_{\{X_n=y\}}\right]$ from Lecture 18. Show that if x is positive recurrent then $\sum_y \mu_x(y) < \infty$.

(b) Assume that there exists there is one positive recurrent state x, then there is a stationary distribution (proportional to μ_x).

(c) Assume that there exists a stationary distribution π. Show that for any $x \in S$ we have $\pi(x) > 0$. Use this fact to show that there exists a constant $C_x > 0$ so that $\pi(y) = C_x\mu_x(y)$ for all y. Conclude that all states are positive recurrent.

Problem 3 - Results for uncountable Markov Chains

Establish the following facts

(a) Let X_n be a Markov Chain with uncountable state space. Let $A \in \sigma(X_0, \ldots, X_n)$ and $B \in \sigma(X_n, X_{n+1}, \ldots)$. Use the Markov Property to show that for any initial distribution μ,

$$\mathbb{P}_\mu(A \cap B \mid X_n) = \mathbb{P}_\mu(A \mid X_n)\mathbb{P}(B \mid X_n).$$

Thus, the past and future are conditionally independent given the present.

(b) (Lévy's 0-1 law) Let $(\Omega, \mathcal{F}, \{cF_n\}, \mathbb{P}\}$ be a filtered space and $\mathcal{F}_\infty$ be the minimal sigma algebra generated by $\{\mathcal{F}_n\}$ and let $X \in L^1$. Show that

$$\mathbb{E}[X \mid \mathcal{F}_n] \to \mathbb{E}[X \mid \mathcal{F}_\infty]$$

both a.s., and in L^1.

(c) Let X_n be a Markov Chain with uncountable state space. Use the previous result to show that if

$$\mathbb{P}\left(\cup_{m=n+1}^{\infty}\{X_m \in B_m\} \mid X_n\right) \geq \delta > 0 \quad \text{on } \{X_n \in A_n\},$$

then, $\mathbb{P}\left(\{X_n \in A_n \text{ i.o.}\} - \{X_n \in B_n \text{ i.o.}\}\right) = 0$.

Problem 4 - Riffle shuffle

Riffle shuffle is the standard way card decks are shuffled at casinos. This shuffling strategy was analyzed by Bayer and Diaconis in the beautiful paper "Trailing the Dovetail Shuffle to its Lair."[1] One can understand the shuffling process as a Markov Chain where the state space is the set of all permutations of $n = 52$ symbols. Casinos would like to be able to start games with decks that are distributed uniformly at random. Fortunately, the uniform distribution is the stationary distribution of this Markov Chain, and Bayer and Diaconis showed that $3/2 \log_2 n$ are necessary and sufficient to have convergence. We will test this claim via simulations.

(a) Read the first two paragraphs of Bayer and Diaconis' paper to understand the riffle shuffle and implement a function that runs k iterations of this Markov Chain initialized from the identity permutation, i.e., $\sigma(1) = 1, \ldots, \sigma(52) = 52$.

(b) Let X_k be the position of the first card after k iterations. Implement a script that estimates the probability distributions of X_k for $k = 1, \ldots, 10$ using a Monte Carlo simulation with 5000 draws of the chain. Let $\widehat{p}_1(\cdot), \ldots \widehat{p}_{10}(\cdot)$ be the distributions you estimated.

(c) Let U be the uniform distribution over $\{1, \ldots, 52\}$. Plot the TV distance $\|\widehat{p}_k - U\|_{\text{TV}}$ against k. How long does it take for the distance to get small? Include your plot!

[1] I highly recommend reading this paper.