Probability Theory 2, Spring 2024 - Homework 2
Due one hour before lecture on 2/26 (Gradescope)

Your submitted solutions to assignments should be your own work. While discussing homework problems with peers is permitted, the final work and implementation of any discussed ideas must be executed solely by you. Acknowledge any source you consult.

Problem 1 - Conditional variance

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a random variable X such that $\mathbb{E}X^2 < \infty$. Define $\text{var}(X \mid \mathcal{F}) = \mathbb{E}(X^2 \mid \mathcal{F}) - \mathbb{E}(X \mid \mathcal{F})^2$.

(a) Show that the following identity holds true:

$$\text{var}(X) = \mathbb{E}(\text{var}(X \mid \mathcal{F})) + \text{var}(\mathbb{E}(X \mid \mathcal{F})).$$

(b) Let $Y_1, Y_2, \ldots$ be i.i.d. random variables with mean μ and variance σ^2, and let N be an integer valued random variable indepedent from the sequence of Y's such that $\mathbb{E}N^2 < \infty$. Define $X = Y_1 + \ldots Y_N$, and show that $\text{var}(X) = \sigma^2 \mathbb{E}N + \mu^2 \,\text{var}(N)$.

Problem 3 - Missing conditional results

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, prove the following two results. **Hint:** Try to generalize the proofs you covered in Probability Theory 1 in the case of the standard expectation.

(a) Use the Conditional Monotone Convergence Theorem that we proved in Lecture 6 to prove the following version Conditional Fatou's Lemma. Let $(X_n)_n$ be a sequence of random variables with $X_n \geq 0$ and $\mathbb{E}|X_n| < \infty$ for all n and let $\mathcal{G}$ be a sub-σ-algebra. Then,

$$\mathbb{E}\left[\liminf_{n\to\infty} X_n \mid \mathcal{G}\right] \leq \liminf_{n\to\infty} \mathbb{E}\left[X_n \mid \mathcal{G}\right].$$

(b) Use Conditional Fatou's to prove the following version of Conditional Dominated Convergence. Let V be a random variable with $\mathbb{E}V \leq \infty$, let $(X_n)_n$ be a sequence of random variables such that $|X_n| \leq V$ and $X_n \to X$ almost surely for some X, and let $\mathcal{G}$ be a sub-σ-algebra. Then,

$$\mathbb{E}[X_n \mid \mathcal{G}] \to \mathbb{E}[X \mid \mathcal{G}] \quad \text{almost surely.}$$

Problem 3 - Funky martingales

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, show the following.

(a) Let $X_1, X_2, \ldots$ be i.i.d. random variables with distribution given by

$$\mathbb{P}(X_1 = 1) = p, \quad \text{and} \quad \mathbb{P}(X_1 = -1) = q = (1-p).$$

and consider their natural filtration $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ (with $\mathcal{F}_0 = \{\emptyset, \Omega\}$). Define $S_n = \sum_{k=1}^n X_k$. Prove that $M_n = (q/p)^{S_n}$ is a martingale with respect to the natural filtration.

(b) Assume the context of the previous question. For any given $\lambda > 0$ determine a scalar C_λ such that

$$Z_n = C_\lambda^n \lambda^{S_n}$$

is a martingale with respect to the natural filtration.

(c) Give an example of a martingale X_n with $X_n \to -\infty$ almost surely. **Hint:** Consider $X_n = \xi_1 + \cdots + \xi_n$ where the ξ_i's are independent, but not indentically distributed, and $\mathbb{E}\xi_i = 0$. Borel-Cantelli will be useful here.

Problem 4 - Stopping times

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, prove the next two results.

(a) Let T, S be stopping times with respect to $(\Omega, \{\mathcal{F}_n\}_n, \mathbb{P})$. Show that $S \wedge T := \min\{S, T\}$, $S \vee T := \max S, T$, and $S + T$ are also stopping times.

(b) Let S and T be stopping times with $S \leq T$. Define the process $\mathbf{1}_{(S,T],n}$ given by

$$\mathbf{1}_{(s,T],n}(\omega) := \begin{cases} 1 & \text{if } S(\omega) < n \leq T(\omega) \\ 0 & \text{otherwise.} \end{cases}$$

Prove that $\mathbf{1}_{(S,T]}$ is previsable, and deduce that if X_n is a supermartingale, then

$$\mathbb{E}(X_{T\wedge n}) \leq \mathbb{E}(X_{S\wedge n}).$$