Lecture 9

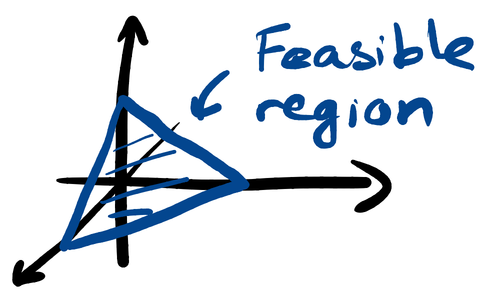
Last time
- ▷ Subdifferential calculus
- ▷ Lagrange duality
- ▷ Fenchel biconjugation

Today
- ▷ Linear programming revisited
- ▷ Extreme points
- ▷ Intro to Simplex.

Linear Programming Revisited

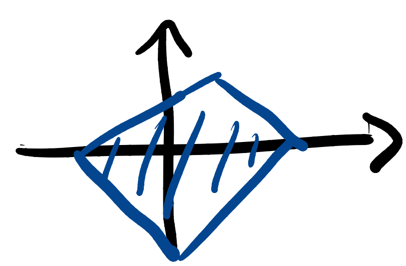
We can always write any LP in "standard form", i.e.,

$$(P) \quad p^* = \begin{cases} \inf & \langle c, x\rangle \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{cases} \qquad A \in \mathbb{R}^{m \times n}$$



From Lecture 7 we know its dual is equal to

$$(D) \quad d^* = \begin{cases} \sup & \langle b, y\rangle \\ \text{s.t.} & A^T y \leq 0. \end{cases}$$



For general conic programming one requires a constraint quali fication condition for strong duality. LPs do not require that.

Proposition: There are exactly 4 possibilities for LPs:

1) Both primal and dual are achieved and $p^* = d^*$.
2) The primal is feasible and the dual infeasible $p^* = -\infty = d^*$.
3) The dual is feasible and the dual infeasible.
4) Both primal and dual are infeasible $p^* = \infty$ and $d^* = -\infty$.

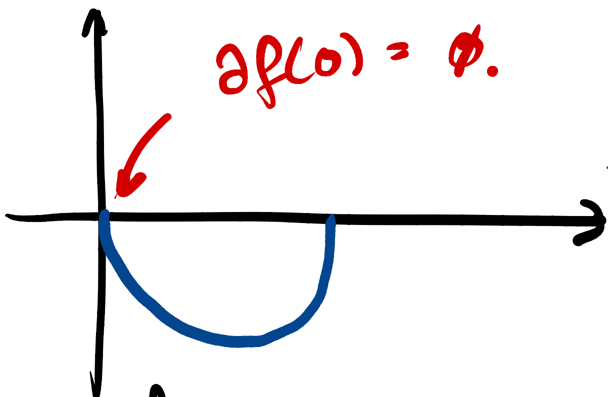
$P = \{x \mid Ax = b, x \geq 0\} = \emptyset$.

Proof: Exercise. $\square$

Intuition

The reason we neededed
$$0 \in \text{int}\{\text{dom } g - A \text{ dom } f\} = \text{int dom } \nu.$$
Value function.

was to guarantee $\exists y \in \partial \nu(0)$. If this doesn't happen a subgradient might fail to exist, e.g.,



$$f(x) = \begin{cases} -\sqrt{1-(x-1)^2} & x \in [0,2] \\ +\infty & \text{otherwise} \end{cases}$$

Closed, convex functions that are piecewise linear on their domain do not have this issue.

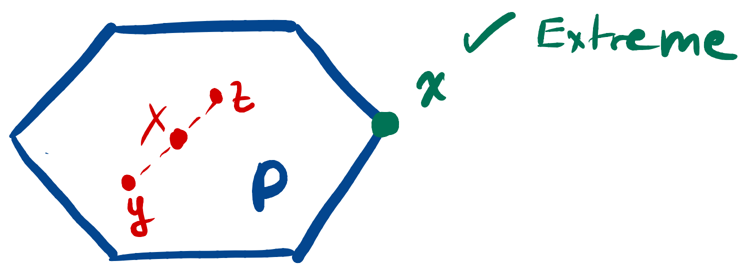
Extreme points

Vertices will play a critical role for simplex, so let's try to understand them better.

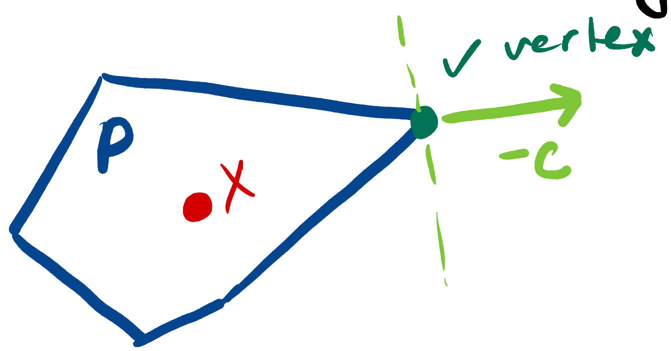
Let
$$P = \{ x \mid Ax \leq b \}$$
be a generic polyhedron.

(when P is bounded we call it polytope.)

Def: We say $x \in P$ is an extreme point if there are no pair of points $y, z \in P$ and $\lambda \in (0,1)$ s.t. $x = \lambda y + (1-\lambda) z$. $\dashv$

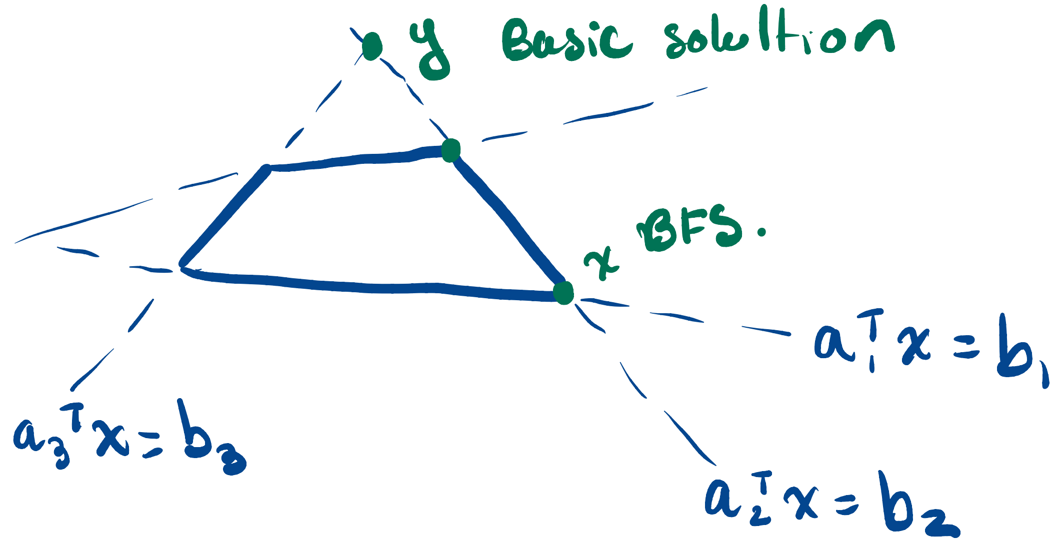


Def: We say $x \in P$ is a vertex if $\exists\, c$ s.t. $c^T x < c^T y \quad \forall y \in P \setminus \{x\}$. $\dashv$



Def: We say $x \in P$ is a Basic Feasible Solution (BFS) if there exist n linearly independent a_i with $a_i^T x = b_i$. $\dashv$

(If we drop the constraint $x \in P$, we say it is a Basic Solution.)



Theorem: The set of extreme points, vertices, and BFS are the equal.

Proof: (Vertices $\subseteq$ Extreme points)

Let $x \in P$ a vertex with witness c.

Take $y, z \in P \setminus \{x\}$ and $\lambda \in (0,1)$, then

$$\begin{array}{rl} & \lambda \, (c^T x < c^T y) \\ + & (1-\lambda)(c^T x < c^T z) \\ \hline & c^T x < c^T(\lambda y + (1-\lambda) z) \end{array}$$

So $x \neq \lambda y + (1-\lambda) z$.

(Extreme points $\subseteq$ BFS)

Suppose $x \in P$ is not a BFS.

Let $I = \{ i \in [m] \mid a_i^T x = b \}$ and so $\{a_i\}_I$ are linearly dependent.

Hence they are contained in a subspace $S = \{ z \in \mathbb{R}^n \mid d^T z = 0 \}$ for some $d \neq 0$. Then for any $\varepsilon > 0$

$$\left.\begin{aligned} a_i^T (x + \varepsilon d) &= b_i \\ a_i^T (x - \varepsilon d) &= b_i \end{aligned}\right\} \quad \forall i \in I.$$

For $i \in I^c$ we can take ε small to ensure

$$a_i^T (x \pm \varepsilon d) \leq b_i \qquad \forall i \in I^c.$$

Thus $x = \frac{1}{2}(x - \varepsilon d) + \frac{1}{2}(x + \varepsilon d)$ is not extreme.

(BFS $\subseteq$ Vertices)

Let $x \in P$ be a BFS and let $I = \{ i \mid a_i^T x = b_i \}$ and let $\bar{I}$ be a subset of size n s.t. $\{a_i\}_{i \in \bar{I}}$ are linearly independent. Consider $c = -\sum_{i \in I} a_i$. For any $y \in P \setminus \{x\}$ we have $a_i^T y \leq b_i \;\; \forall i$ and so

$$c^T y = -\sum_{i \in I} a_i^T y > -b_i$$

$$= -\sum_{i \in I} a_i^T x$$

$\exists i$ s.t. $a_i y < b$ as otherwise $y = x$.

$$= c^T x. \qquad \blacksquare$$

Vertices of standard form Polyhedra.

Recall that a for standard form problems we have

$$P = \{x \mid Ax = b,\ x \geq 0\}.$$

$A \in \mathbb{R}^{m \times n}$

WLOG we may assume the rows of A are independent (why?)

Thus for any BFS we will have n linearly independent constraint hold tightly:

- m come from $Ax = b$.
- $n-m$ from nonnegativity constraints s.t. $x_i = 0$.

Def: We call any $B = \{B_1, \dots B_m\} \subseteq [n]$ of size m. $\dashv$

For any vector $x \in \mathbb{R}^n$ and a set $S = \{s_1, \dots, s_k\} \subseteq [n]$, let

$$x_S = (x_{s_1}, \dots, x_{s_k}).$$

For $A \in \mathbb{R}^{m \times n}$, let

$$A_S = \begin{pmatrix} | & & | \\ A_{s_1} & \cdots & A'_{s_k} \\ | & & | \end{pmatrix}$$

↖ rows of A.

Note that any BFS $x \in P$ corresponding to a basis B is the unique solution of

$$\begin{cases} Ax = b \\ x_{B^c} = 0 \end{cases} \iff \begin{cases} A_B x_B + A_{B^c} x_{B^c} = b \\ x_{B^c} = 0 \end{cases}$$

$$\iff \begin{cases} x_B = A_B^{-1} b \\ x_{B^c} = 0. \end{cases} \qquad (B)$$

Lemma ✓: Any nonempty P in standard form has at least one BFS.

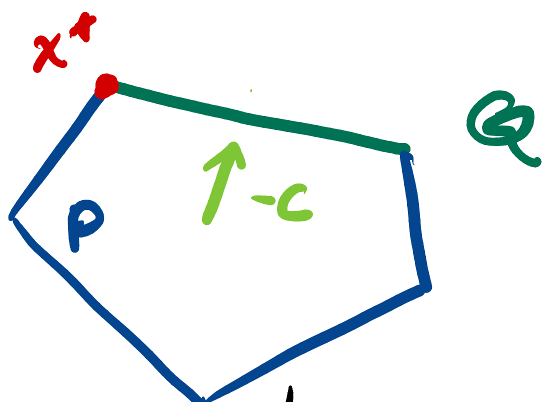
(P cannot have infinite lines $\{x_0 + \lambda v \mid \lambda \in \mathbb{R}\}$)

Proof: Exercise. $\square$

Theorem: If (P) achives a minimizer, then some BFS is a minimizer.

Proof: Let $Q = \left\{ x \;\middle|\; \begin{array}{l} Ax = b,\ x \geq 0 \\ c^T x = p^* \end{array} \right\}$ be the set of minimizers.

Then by Lemma V there is a BFS of Q, call it x^*. Let's show that x^* is also a BFS of P.



Let $y, z \in P$ and $\lambda \in (0,1)$

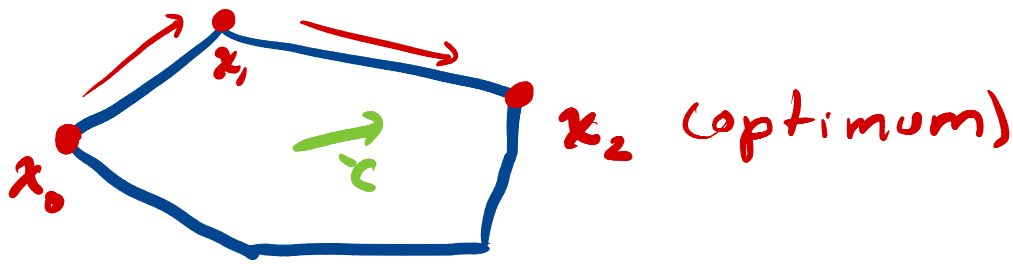
Consider two cases:

- If $y, z \in Q$, then $\lambda y + (1-\lambda) \neq x$ since x is extreme.
- If either y or z belong $P \setminus Q$, say is y. Then
$$c^T(\lambda y + (1-\lambda) z) = \lambda c^T y + (1-\lambda) c^T z$$
$$y \in P \setminus Q \longrightarrow \quad > c^T x.$$

Thus $x \neq \lambda y + (1-\lambda) z$. $\square$

The simplex method.

This suggests a simple strategy: start at a vertex and move to "neighboring vertices" that improve function value:



We do this using bases. Let $x(B)$ be the solution of (♡).

SIMPLEX (Informal)

▷ Pick a basis B_0 s.t. $x(B_0)$ is feasible.

▷ Loop $k \geq 0$:

$i \in B$ $\quad$ $j \notin B_k$

- ▷ Update $B_{k+1} \leftarrow B_k \setminus \{i\} \cup \{j\}$ s.t.
  1. $x(B_{k+1})$ is feasible
  2. $c^T x(B_{k+1}) \leq c^T x(B_k)$.
- ▷ If $x(B_{k+1})$ is optimal: Return $x(B_{k+1})$.

How to guarantee these?