Lecture 23

Last time
- Limiting subdifferential
- Calculus rules

Today
- Clarke subdifferential
- Basic Measure Theory

Clarke subdifferential

One awkward issue with the limiting subdifferential is that it doesn't yield convex sets. Recall

Def: For any set $S \subseteq E$

$$\text{conv}\, S = \left\{ \sum_i \lambda_i x_i \;\middle|\; x_i \in E, \lambda_i > 0, \sum_i \lambda_i = 1 \right\}$$

Caratheodory Theorem:
$$= \left\{ \sum_{i=1}^{\dim E+1} \lambda_i x_i \;\middle|\; x_i \in E, \lambda_i > 0, \sum_{i=1}^{\dim E+1} \lambda_i = 1 \right\}$$
⊣

Proposition: If S is compact, so is conv S. ⊣

Def (Clarke subdifferential): For a locally Lipschitz function $f: E \to \mathbb{R} \cup \{+\infty\}$ define

$$\partial_c f(x) = \text{conv}\, \partial f_L(x).$$
⊣

Remark: Clarke introduced this definition and developed theory for it in his PhD thesis in 1973.
This subdifferential is often used in to establish algorithmic convergence. ⊣
We will establish an intuitive characterization.

Interlude: Basic facts from Measure Theory

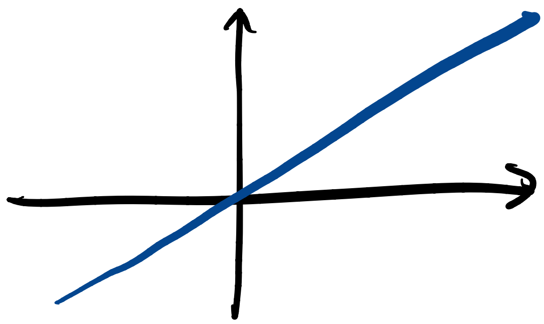
Def: Let $S \subseteq \mathbb{R}^d$. We say that this set has **measure zero** if for all $\varepsilon > 0$ there is a countable sequence of boxes $B_1, B_2, \ldots$ with $S \subseteq \bigcup_{i=1}^{\infty} B_i$ s.t. $\sum_i \text{vol } B_i \leq \varepsilon$.

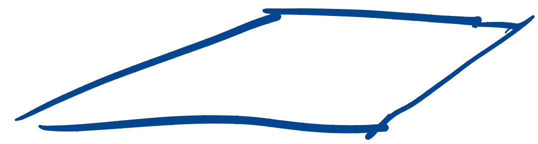
We say that a property holds **almost everywhere (a.e.)** if it holds in the complement of a measure zero set. ⊣

Examples

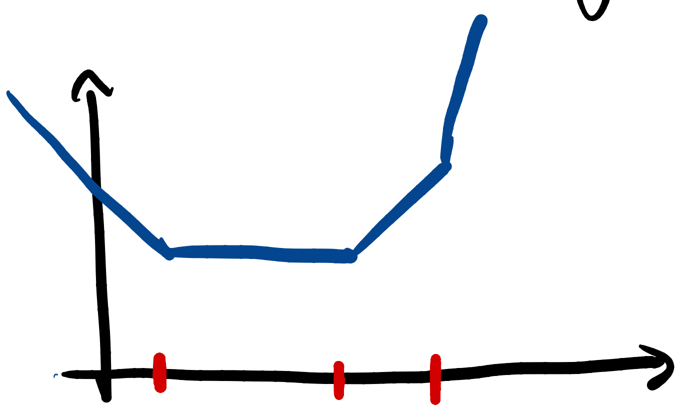
- A countable set has measure zero

- Any affine subspace $S \subseteq E$ with $\dim E > \dim S$ has measure zero.





- Polyhedral functions are differentiable almost everywhere.



This last example is not just a happy coincidence.

Theorem (Rademacher's Theorem): Locally lipschitz functions are differentiable almost everywhere.

We will use one more classical result.

Theorem (Fubini's Theorem): Suppose that $S \subseteq E$ has measure zero, then for almost all x, the set

$$\{ t \in \mathbb{R} \mid x + tz \}$$

has measure zero for all z. ⊣

Back to Clarke

Theorem: Let $f: E \to \mathbb{R}$ be Lipschitz and $S \subseteq E$ be the set where f is not differentiable. Then

$$\partial_c f(x) = \operatorname{conv} \underbrace{\{ \lim_{n \to \infty} \nabla f(x_n) \mid x_n \to x,\ x_n \notin S \}}_{\tilde{\partial} f(x)}.$$

Exercise: $\tilde{\partial} f(x)$ is nonempty and compact.

Proof: We start with a claim:

Claim (□→): $\partial f(x) \subseteq \tilde{\partial} f(x)$. (↙ Frechet)

Before proving this claim, let

us see how it implies the result. By the claim $\text{grph}\, \partial f \subseteq \text{graph}\, \tilde{\partial} f$. It is easy to show (do it!) that $\text{graph}\, \tilde{\partial} f$ is closed. By taking the closure of $\text{grph}\, \partial f$, we obtain that $\text{grph}\, \partial_L f \subseteq \text{grph}\, \tilde{\partial} f$. Hence $\partial_L f(x) \subseteq \tilde{\partial} f(x)$ and so

$$\text{conv}\, \partial_L f(x) \subseteq \tilde{\partial} f(x).$$

For the opposite direction, notice that if f is differentiable at $x_n \Rightarrow \{\nabla f(x_n)\} = \partial f(x)$ and so

$$\text{conv}\, \partial_L f(x) \supseteq \tilde{\partial} f(x).$$

Proof of Claim (☐—): Suppose seeking contradiction that $\exists\, y \in \partial f(x) \setminus \tilde{\partial} f(x)$. Since $\tilde{\partial} f(x)$ is convex and closed, there is $z \in E$, $\varepsilon > 0$ s.t.

$$\langle y, z \rangle \geq \max_{g \in \tilde{\partial} f(x)} \langle g, z \rangle + 2\varepsilon.$$

So for points $\tilde{x} \in S^c$ close to x

$(\circledast)$ $\langle y, z \rangle \geq \langle \nabla f(\tilde{x}), z \rangle + \varepsilon.$ (Why?)

By definition $f(x+tz) - f(x) \geq t\langle y, z \rangle + o(t).$

Thus, for small $t > 0$

$$\frac{1}{t}\left(f(x+tz) - f(x)\right) \geq \langle y, z \rangle - \frac{\varepsilon}{3}$$

By Fubini's there is a point $\bar{x}$ arbitrarily close to x s.t. $\{t \mid \bar{x} + tz \in S\}$ has measure zero. Choose $\bar{x}$ so that

$$\frac{1}{t}\left(f(\bar{x}+tz) - f(\bar{x})\right) \geq \langle y, z \rangle - \frac{2\varepsilon}{3}, \quad (=)$$

which can be done since everything is continuous in x.

Now by the fundamental theorem of calculus

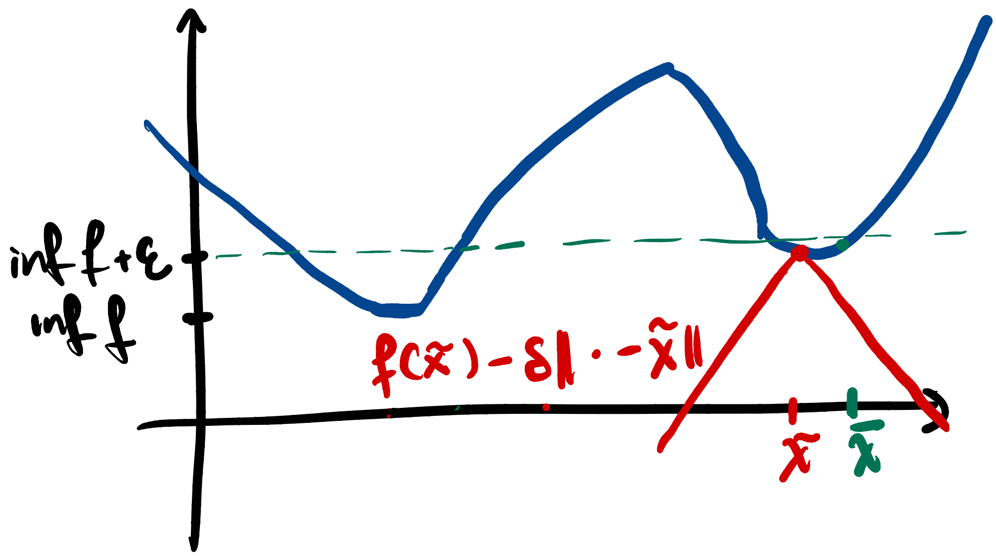
$$f(\bar{x}+tz) - f(\bar{x}) = \int_0^t \langle \nabla f(\bar{x}+\tau z), z \rangle \, d\tau$$

$$\leq t(\langle y, z \rangle + \varepsilon),$$

which contradicts (==). ☐

This concludes the proof of the theorem. ☐

Intuition



Proof: We want to find a conic minorant as in our intuition. Define the function

$$c(x) = f(\bar{x}) - \delta \|x - \bar{x}\|.$$

If $f(x) > c(x)$ $\forall x \neq x$, then, we can take $\tilde{x} = x$. Suppose that this is not the case. Define

$$
\begin{aligned}
W &:= \{ x \mid f(x) \le c(x) \} \\
&= \{ x \mid f(x) + \delta \|x - \bar{x}\| \le f(\bar{x}) \}. \\
&\subseteq \{ x \mid \delta \|x - \bar{x}\| \le f(\bar{x}) - \inf f \} \\
&\subseteq \{ x \mid \|x - \bar{x}\| \le \varepsilon/\delta \}
\end{aligned}
$$

Notice that W is nonempty, bounded

and closed. ($\leftarrow$ since $f + \delta\|\cdot - \bar{x}\|$ is closed) Therefore, the function $f(x) + \iota_W(x)$ achieves a minimizer $\tilde{x}$. Since $\bar{x} \in W$, we have that

(☺) $$f(\tilde{x}) + \delta\|\tilde{x} - \bar{x}\| \le f(\bar{x}).$$

It is then immediate that **1.** and **2.** hold. Moreover, for $x \in W \setminus \tilde{x}$ we have $f(\tilde{x}) \le f(x)$ and so

$$f(\tilde{x}) + 0 < f(x) + \delta\|x - \tilde{x}\|.$$

For $x \in W^c$, then

$$\begin{aligned} f(\tilde{x}) &\overset{(☺)}{\le} f(\bar{x}) - \delta\|\tilde{x} - \bar{x}\| \\ &\overset{x \in W^c}{<} f(x) + \delta\|x - \bar{x}\| - \delta\|\tilde{x} - \bar{x}\| \\ &\le f(x) - \delta\|x - \tilde{x}\| \end{aligned}$$

where the last line follows by the reverse triangle inequality. Thus **3.** follows.

□

Mean Value Theorem

Next we will cover a applications of these subdifferentials and their calculus rules.

Theorem (Mean value 1) Consider a proper closed function $f: E \to \mathbb{R} \cup \{\infty\}$, and fix two points $x_0, x_1 \in \operatorname{dom} f$. Then, for $\forall \varepsilon > 0$, there exists a subgradient $v \in \partial f(x)$ with $x \in [x_0, x_1] + \varepsilon \mathbb{B}$

$$f(x_1) - f(x_0) \leq \langle v, x_1 - x_0 \rangle + \varepsilon.$$

Proof: Consider the map

$$c(t) = \begin{pmatrix} \overbrace{x_0 + t(x_1 - x_0)}^{c_1(t)} \\ t \\ t \end{pmatrix} \in \begin{pmatrix} E \\ \mathbb{R} \\ \mathbb{R} \end{pmatrix}$$

and

$$h(y, t_1, t_2) = f(y) - f(x_0) - t_1(f(x_1) - f(x_0)) + \iota_{[0,1]}(t_2)$$

Then, the function

$$\varphi(t) = h(c(t)) = f(c_1(t)) - f(x_0) - t(f(x_1) - f(x_0)) + \iota_{[0,1]}(t)$$

Since this function is proper closed proper and bounded, it admits a minimizer $\hat{t} \in [0,1]$. Applying the fuzzy sum rule we get $\exists\, t_1 \in \mathbb{R}$, $t_2 \in [0,1]$, $x \in E$ s.t.

$$\max\{|t_1 - \hat{t}|, |t_2 - \hat{t}|, \|x - c(\hat{t})\|, |f(x) - f(c(\hat{t}))|\} \leq \varepsilon.$$

and

$$0 \in \langle \partial f(x), x_1 - x_0 \rangle - (f(x_1) - f(x_0)) + N_{[0,1]}(t_2) + [-\varepsilon, \varepsilon]. \qquad (\heartsuit)$$

We consider two cases

- **Case 1:** $\hat{t} < 1$. In that case $(\heartsuit)$ reduces to $\exists\, v \in \partial f(x)$ and $s \in \mathbb{R}_-$ s.t.

$$f(x_1) - f(x_0) \leq \langle v, x_1 - x_0 \rangle + s + q$$ ($q \in [-\varepsilon, \varepsilon]$)

$\leq \langle v, x_1 - x_0 \rangle + \varepsilon.$

▷ Case 2 : $\hat{t} = 1$. Then, noticed that $\varphi(1) = \varphi(0) = 0$ and so 0 is also a minimizer and we can fold back to case 1. □