Nonlinear Optimization 2, Spring 2025 - Homework 5
Due at 11:49PM on Friday 5/2 (Gradescope)

Your submitted solutions to assignments should be your own work. You are allowed to discuss homework problems with other students, but should carry out the execution of any thoughts/directions discussed independently, on your own. Acknowledge any source you consult. Do not use any type of Large Language Model, e.g., ChatGPT, to blindly answer this assignment. If you do, your submission will be voided and you will get zero as a grade.

Problem 1 - The good old days

Prove that the Frechet subdifferential (let us call it $\partial_F f$ for the purposes of this assignment) coincides with the convex subdifferential ∂f when the function f is convex, and with the gradient $\{\nabla f\}$ when f is continuously differentiable.

Problem 2 - A simple example

Compute the Frechet and limiting subdifferential for the function $f\colon \mathbf{R}^2 \to \mathbf{R}$ defined via $f(u, v) = |u| - |v|$.

Problem 3 - Collect all the properties

Suppose that the function $f\colon \to \mathbf{R} \cup \{+\infty\}$ is finite at $\overline{x}$.

(a) If $\overline{x}$ is a local minimizer, prove that $0 \in \partial_F f(\overline{x})$.

(b) If $0 \in \partial_F f(\overline{x})$ and $\delta > 0$, prove that $\overline{x}$ is a strict local minimizer of the function $x \mapsto f(x) + \delta\|x - \overline{x}\|$.

Now, assume that f is closed around $\overline{x}$.

(c) If $(x_n, f(x_n), y_n) \to (\overline{x}, f(\overline{x}), \overline{y})$ with $y_n \in \partial_L f(x_n)$, prove $\overline{y} \in \partial_L f(\overline{x})$.

(d) Use the Fuzzy sum rule to give another proof of the density theorem, i.e., $\partial_F f(x)$ is not empty at points arbitrarily close to $\overline{x}$.

Finally, suppose further that f is Lispchitz around $\overline{x}$.

(e) Prove that $\partial_L f(\bar{x})$ is nonempty.

(f) Prove that $\partial_L f(\bar{x})$ is compact.

(g) Prove that $\partial_c f(\bar{x})$ is also nonempty and compact.

Problem 4 - So close to having a critical point

Suppose that a closed function $f\colon \mathbf{E} \to \mathbf{R} \cup \{+\infty\}$ is bounded from below. Prove that zero lies in the closure of the set

$$\bigcup_{x \in \mathbf{E}} \partial_F f(x).$$