# Nonlinear Optimization 2, Spring 2025 - Homework 4
# Due at 11:49PM on Friday 4/18 (Gradescope)

**Your submitted solutions to assignments should be your own work. You are allowed to discuss homework problems with other students, but should carry out the execution of any thoughts/directions discussed independently, on your own. Acknowledge any source you consult. Do not use any type of Large Language Model, e.g., ChatGPT, to blindly answer this assignment. If you do, your submission will be voided and you will get zero as a grade.**

## Problem 1 - Polyhedra again

Consider a polyhedral function $f: \mathbf{E} \to \mathbf{R} \cup \{+\infty\}$, i.e., its epigraph is a polyhedron. Suppose we generate a sequence via

$$x_{k+1} \leftarrow \operatorname{prox}_f(x_k).$$

Show that there is a finite number of polyhedra $P_1, \ldots, P_k$ such that for any $x \in \mathbf{E}$ there is $i \in [k]$ such that $\partial f(x) = P_i$. Use this fact to show that the sequence $x_k$ converges to a minimizer after finitely many steps.

## Problem 2 - Pythagoras' dream

Let $f: \mathbf{E} \to \mathbf{R} \cup \{+\infty\}$ be a closed, convex, proper function and $z \in \mathbf{E}$ arbitrary.

**(a)** Show that

$$\frac{1}{2}\|z\|^2 = \inf_{x \in \mathbf{E}} \left\{ f(x) + \frac{1}{2}\|x - z\|^2 \right\} + \inf_{y \in \mathbf{E}} \left\{ f^*(y) + \frac{1}{2}\|y - z\|^2 \right\}.$$

**(b)** Prove that the optimal solutions $x^\star, y^\star$ of the two problems above are attained and can be characterized by

$$z = x^\star + y^\star \quad \text{and} \quad y^\star \in \partial f(x^\star).$$

**(c)** Conclude that $y^\star \in \partial f(x^\star)$ if, and only if, $x^\star \in \partial f^*(y^\star)$.

## Problem 3 - Le wild linear system appears

Consider the problem of minimizing

$$\inf_{x \in \mathbf{E}} f(x) + g(Ax)$$

with $f: \mathbf{E} \to \mathbf{R} \cup \{+\infty\}$ and $g: \mathbf{Y} \to \mathbf{R} \cup \{+\infty\}$ be closed, convex, proper functions, and $A: \mathbf{E} \to \mathbf{Y}$ be a linear map.

**(a)** Rewrite this problem as the ADMM problem we considered in Lecture 18 with two variables $x$ and $z$. Write down the ADMM algorithm. Can you write this only in terms of $\operatorname{prox}_{\alpha f}$ and $\operatorname{prox}_{\alpha g}$?

**(b)** Consider a different reformulation of the form

$$\inf_{y,w,z} \{f(y) + g(z) \mid w = y, Aw = z\}.$$

Write down the ADMM update using variables $x = (y, w)$ and $z$. Do you need to solve a linear system?

**(c)** Let $A \in \mathbf{R}^{m\times n}$, $b \in \mathbf{R}^m$, and $c \in \mathbf{R}^n$. Consider the linear programming problem $\inf\{c^\top x \mid Ax = b, x \geq 0\}$. Let $f(x) = c^\top x + \iota_{\{w \mid Aw=b\}}(x)$ and $g(x) = \iota_{\mathbf{R}^n_+}(x)$, and write the ADMM update for the problem $\inf\{f(x) + g(z) \mid x = z\}$. Do you need to solve a linear system?

## Problem 4 - Matrix-free linear programming

**(a)** Given a matrix $A \in \mathbf{R}^{m\times n}$ and vectors $c \in \mathbf{R}^n, b \in \mathbf{R}^m$. Consider the problem of minimizing

$$\inf\{c^\top x \mid Ax = b, x \geq 0\}.$$

Consider the functions $f(x) = c^\top x + \iota_{\mathbf{R}^n_m}(x)$ and $g(w) = \iota_{\{b\}}(w)$. Find explicit expressions for $\mathbf{prox}_{\tau f}$ and $\mathbf{prox}_{sg^*}$.

**(b)** Code the PDHG method from scratch based on the pseudocode from Lecture 19. You can only use Numpy or SciPy for matrix operations, you cannot call a method that already implements PDHG.

**(c)** Denote the set of all feasible grading rubrics $(H, M, F) \in \mathbb{R}^3$ as

$$\mathcal{Q} = \{(H, M, F) \mid H{+}M{+}F \leq 100,\ H, M \geq 15,\ F \geq M,\ 50 \leq M{+}F \leq 80,\ H{+}M{+}F \geq 90\}.$$

Recall that we will compute your grade as maximization problem of the form $\max\{b^\top y : A^\top y \leq c\}$, write the $A, b$, and $c$. How shall you set stepsizes $\tau$ and $s$ in PDHG to ensure convergence for this problem?

**(c)** Use your PDHG implementation to find the grades of the following hypothetical students:

$$(C_H, C_M, C_F, C_P) = (100, 90, 80, 70),$$

$$(C_H, C_M, C_F, C_P) = (85, 85, 85, 85),$$

$$(C_H, C_M, C_F, C_P) = (70, 80, 90, 100),$$

$$(C_H, C_M, C_F, C_P) = (90, 95, 85, 80).$$