Nonlinear Optimization 2, Spring 2025 - Homework 2
Due at 11:49PM on Friday 2/28 (Gradescope)

Your submitted solutions to assignments should be your own work. You are allowed to discuss homework problems with other students, but should carry out the execution of any thoughts/directions discussed independently, on your own. Acknowledge any source you consult. Do not use any type of Large Language Model, e.g., ChatGPT, to blindly answer this assignment. If you do, your submission will be voided and you will get zero as a grade.

Problem 1 - Directional derivative formulae

(a) Let $f\colon \mathbf{E} \to \mathbf{R}$ be continuous and directionally differentiable at zero such that there exists $g \in \mathbf{E}$ satisfying $f'(x; v) = \langle g, v\rangle$ for all $v \in \mathbf{E}$. Find a counterexample to show that this does not necessarily imply that f is differentiable at zero.

(b) Let $f\colon \mathbf{E} \to \mathbf{R} \cup \{+\infty\}$ be a proper, closed, convex function. Let $x \in \operatorname{dom} f$, show that for any v, the directional derivative $f'(x, v)$ exists and, moreover, it is equal to

$$f'(x; v) = \sup_{g\in\partial f(x)} \langle g, v\rangle \quad \text{for all } x, v \in \mathbf{E}.$$

(c) Let $f_1, \ldots, f_k\colon \mathbf{E} \to \mathbf{R}$ be differentiable functions and define $h(x) = \max_{j\in[k]} f_j(x)$. Prove that h is directionally differentiable for any $v \in \mathbf{E}$. Further, prove that

$$h'(x; v) = \max_{j\in M(x)} \langle \nabla f_j(x), v\rangle \quad \text{for all } x, v \in \mathbf{E},$$

where $M(x) = \{i \in [k] \mid f_i(x) = h(x)\}$.

Problem 2 - Arguments that we missed

Show the following two things we did not prove in class.

(a) Let $f\colon \mathbf{E} \to \mathbf{R} \cup \{+\infty\}$ and $g\colon \mathbf{E} \to \mathbf{R} \cup \{+\infty\}$ be convex functions, and $A\colon \mathbf{E} \to \mathbf{Y}$ be a linear map. Define the value function $\nu\colon \mathbf{Y} \to \mathbf{R} \cup \{\pm\infty\}$ given by $\nu(z) = \inf_{x\in\mathbf{E}} f(x) + g(Ax + z)$.

  (1) Show that ν is a convex function.

  (2) In Lecture 6, we concluded that if $v(0)$ is finite and $0 \in \operatorname{int}\{\mathrm{dom(g)} - \mathrm{A\,dom(f)}\}$, then there exists a $y \in \partial\nu(0)$. To do this we used the "Existance of subgradients" Theorem from Lecture 4. However, this theorem only applies to functions whose image land in $\mathbf{R} \cup \{+\infty\}$, why can we apply it to ν? (Recall that we saw an example where $\nu(1) = -\infty$ in Lecture 7).

(b) Let $a_1, \ldots, a_m \in \mathbf{E}$ be an arbitrary collection of points. Consider the following three statements.

  (1) The function $f(x) = \log\left(\sum_{i\in[m]} \exp\left(\langle a_i, x\rangle\right)\right)$ is bounded below.

(2) There exists a vector $\lambda \in \mathbf{R}^m_+$ such that $\sum_{i\in[m]} \lambda_i = 1$ and $\sum_{i\in[m]} \lambda_i a_i = 0$.

(3) There is no vector $x \in \mathbf{E}$ such that $\langle a_i, x\rangle < 0$ for all $i \in [m]$.

In class we proved that (1) implies (2). Show that (2) implies (3) and (3) implies (1).

Problem 3 - Duality with cones

(a) (**Krein-Rutman Theorem**) Consider a linear map $A\colon \mathbf{E} \to \mathbf{Y}$, and the indicator functions of convex cones $K \subseteq \mathbf{E}$ and $H \subseteq \mathbf{Y}$. Compute $\partial \iota_K(0)$ and use subdifferential calculus to find conditions guaranteeing $\left(K \cap A^{-1}H\right)^+ = K^+ + A^*H^+$ where $A^{-1}H = \{x \in \mathbf{E} \mid Ax \in H\}$.

(b) Given a nonempty set $K \subseteq \mathbf{E}$, by considering ι_K^{**}, prove $K = K^{++}$ if, and only if, K is a closed convex cone.

(c) Suppose that the closed convex cones $K \subseteq \mathbf{E}$ and $H \subseteq \mathbf{E}$ satisfy the condition $K^+ \cap \operatorname{int} \mathrm{H}^+ \neq \emptyset$. Prove $K + H$ is closed.

(d) Prove the sum of the closed convex cones in $\mathbf{R}^2 \times \mathbf{R}$

$$\{(x, r) \mid \|x\|_2 \le r\} \quad \text{and} \quad \{(x, r) \mid x_1 = 0, r = x_2\}$$

is not closed. **Hint:** consider the point $-(1, 1, 1)$.

Problem 4 - Von Neumann Minimax Theorem

Suppose the sets $C \subseteq \mathbf{E}$ and $D \subseteq \mathbf{Y}$ are nonempty and convex, with D closed.

(a) By considering the Fenchel problem

$$\inf_x \{\iota_C(x) + \iota_D^*(Ax)\}$$

prove that if either of the next conditions hold

(i) D is bounded,

(ii) A is surjective and $0 \in \operatorname{int} \mathrm{C}$,

then

$$\inf_{x\in C} \sup_{y\in D} \langle Ax, y\rangle = \sup_{y\in D} \inf_{x\in C} \langle Ax, y\rangle$$

where the supremum on the right is attained whenever finite.

(b) If both C and D are compact, prove

$$\min_{x\in C} \max_{y\in D} \langle Ax, y\rangle = \max_{y\in D} \min_{x\in C} \langle Ax, y\rangle$$

where all the maxima and minima are attained.