3.1 Optimality conditions (Continued)

**Theorem 3.1 (1st order sufficient condition)** Assume $f : \mathbb{R}^d \to \mathbb{R}$ is convex and differentiable. Then $x^*$ is a global minimizer $\iff \nabla f(x^*) = 0$.

**Proof:** “$\Rightarrow$”: Follows from the 1st order necessary condition.

“$\Leftarrow$”: Let $\bar{y} \in \mathbb{R}^d \setminus \{x^*\}$. Define $\psi(t) = f(x^* + t(\bar{y} - x^*))$. By the chain rule $\psi'(0) = \nabla f(x^*)(\bar{y} - x^*) = 0$.

For any $t \in (0, 1]$:

$$
\frac{f(x^* + t(\bar{y} - x^*)) - f(x^*)}{t} \leq \frac{(1 - t)f(x^*) + tf(\bar{y}) - f(x^*)}{t} = f(\bar{y}) - f(x^*).
$$

Thus, by taking the limit as $t$ goes to zero, we get $0 = \psi'(0) \leq f(\bar{y}) - f(x^*)$, and we have that $f(x^*) \leq f(\bar{y})$.

**Theorem 3.2 (2nd order necessary condition)** Suppose that $f : \mathbb{R}^d \to \mathbb{R}$ is twice differentiable. If $x^*$ is a local minimizer, then $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*) \succeq 0$.

Note that $\nabla^2 f(x^*) \geq 0$, means for all $\bar{s} \in \mathbb{R}^d \setminus \{0\}$ we have that $\bar{s}^T \nabla^2 f(x^*) \bar{s} \geq 0$.

**Proof:** Seeking contradiction, assume $\nabla f(x^*) = 0$ and there exists a $\bar{s} \in \mathbb{R}^d \setminus \{0\}$ s.t. $\bar{s}^T \nabla^2 f(x^*) \bar{s} < 0$ and $||\bar{s}|| = 1$.

Define $\psi(t) = f(x^* + t\bar{s})$. Then

$$
0 > \frac{1}{2} \psi''(0) = \lim_{t \to 0} \frac{\psi(t) - \psi(0)}{t^2}.
$$

For small enough $t > 0$, we have that $\frac{\psi(t) - \psi(0)}{t^2} \leq \frac{1}{2} \psi''(0) < 0$. But this means that $f(x^* + t\bar{s}) < f(x^*)$, which is a contradiction.

**Theorem 3.3 (2nd order sufficient condition)** Suppose that $f : \mathbb{R}^d \to \mathbb{R}$ is twice differentiable. We have that $x^*$ is a strict local minimizer if $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*) > 0$.

**Proof:**

Suppose $x^*$ satisfies the assumptions ($\nabla f(x^*) = 0$ and $\nabla^2 f(x^*) > 0$). Take $\bar{U} \in \mathbb{R}^d$ s.t. $||\bar{U}|| = 1$. Let $\psi(t) = f(x^* + t\bar{U})$.

By the Fundamental Theorem of Calculus (FTC), we have that:

$$
\phi(s) = \phi(0) + \int_0^s \phi'(\alpha)d\alpha.
$$
Applying FTC again to $\phi'(\alpha)$:

$$\phi(s) = \phi(0) + \int_0^s \phi'(\alpha) + \int_0^\alpha \phi''(\beta) \, d\beta \, d\alpha. \quad (3.1)$$

Since $\nabla^2 f(\bar{x})$ is continuous and $\lambda_{\min}(\nabla^2 f(\bar{x}^*)) > 0$. For all $\bar{y}$ close to $\bar{x}^*$, we have that $\lambda_{\min}(\nabla^2 f(\bar{y})) \geq \lambda > 0$ where $\lambda$ is some positive constant.

From (3.1) we have that

$$\phi(s) = \phi(0) + \phi'(0)s + \int_0^s \int_0^\alpha \bar{U}^\top \nabla^2 f(\bar{x}^* + \beta \bar{U}) \bar{U} \, d\beta \, d\alpha \geq \phi(0) + \lambda \int_0^s \int_0^\alpha 1 \, d\beta \, d\alpha = f(\bar{x}^*) + \frac{\lambda}{2} s^2.$$

Note that the above theorem is not an if and only if, as can be seen with the following counter example.

**Example 3.4** Let $f(x) = x^4$. Then $x = 0$ is clearly a global minimizer, but $f''(0) = \nabla^2 f(0) = 0$.

![Figure 3.1: Plot of $f(x) = x^4$.](image)

### 3.2 Basic Convexity

**Definition 3.5** A set $C \subseteq \mathbb{R}^d$ is **convex** if for all $\bar{y} \in C$ and $t \in [0,1]$:

$$t\bar{x} + (1-t)\bar{y} \in C.$$

I.e. the straight line between any two points in $C$ must be entirely within $C$. 
Definition 3.6 Given any function $f : \mathbb{R}^d \to \mathbb{R}$ its epigraph is given by:

$$\text{epi} f = \{(\bar{x}, t) \mid f(\bar{x}) \leq t \}.$$ 

I.e. all points that are on or above the graph of $f(x)$.

Figure 3.2: An example of a non-convex set in $\mathbb{R}^2$.

Theorem 3.7 A function $f$ is convex iff the epigraph $\text{epi} f$ is convex.

Proof: Homework exercise. Follows easily from the definitions.

Lemma 3.8 Let $C_1, C_2 \subseteq \mathbb{R}^d$ be convex sets. Then $C_1 \cap C_2$ is also convex.

Proof: Let $x, y \in C_1 \cap C_2$, let $t \in [0, 1]$. Since $C_1$ is convex $t\bar{x} + (1 - t)\bar{y} \in C_1$, and since $C_2$ is convex $t\bar{x} + (1 - t)\bar{y} \in C_2$. Thus $t\bar{x} + (1 - t)\bar{y} \in C_1 \cap C_2$. 

Figure 3.3: The epigraph of $f(x) = x \sin(x) + 1$. 

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