# Mathematics of Data Science, Fall 2025 - Homework 3
# Due at 11:49PM on Sunday Oct/19 (Gradescope)

**Your submitted solutions to assignments should be your own work. You are allowed to discuss homework problems with other students, but should carry out the execution of any thoughts/directions discussed independently, on your own. Acknowledge any source you consult. Do not use any type of Large Language Model, e.g., ChatGPT, to blindly answer this assignment. If you do, your submission will be voided and you will get zero as a grade.**

## Problem 1 - Fun with eigenvalues

Let $A \in \mathbf{R}^{n\times m}$ be a matrix with $n \leq m$. Prove the following.

**(a)** The eigenvalues of $AA^\top$ correspond to $\sigma_1(A)^2, \ldots, \sigma_n(A)^2$ and similarly, $A^\top A$ has egienvalues $\sigma_1(A)^2, \ldots, \sigma_n(A)^2$ together with $m - n$ copies of the eigenvalue zero. Based on this obsevation give an alternate proof of the singular value decomposition theorem from Lecture 9 using the spectral decomposition theorem.

**(b)** Just for this question suppose further that $A \in \mathcal{S}^n$ (i.e., the matrix is symmetric $n \times n$). Somebody lands you the eigenvalue decomposition $A = U\Lambda U^\top$ ($U \in O(n)$ and $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$). Provide an algorithm, and a justification of its correctness, to compute the singular value decomposition of $A$ from its eigenvalue decomposition.

**(c)** Establish the Courant–Fischer min–max formula

$$\sigma_i(A) = \sup_{\substack{V\subset\mathbb{R}^m \\ \dim V = i}} \inf_{\substack{v\in V \\ \|v\|_2=1}} \|Av\|_2, \tag{1}$$

for all $1 \leq i \leq n$, where the supremum ranges over all $i$-dimensional subspaces $V$ of $\mathbb{R}^m$.

**(d)** Use (c) to show the next two inequalities.

1. For all $1 \leq i, j$ with $i + j - 1 \leq n$,

$$\sigma_{i+j-1}(A + B) \leq \sigma_i(A) + \sigma_j(B).$$

2. For all $1 \leq i \leq n$,

$$\left|\sigma_i(A + B) - \sigma_i(A)\right| \leq \|B\|_{\mathrm{op}}.$$

## Problem 2 - Operating at large scale

Let $A \in \mathbf{R}^{n\times m}$ be a matrix with $n \leq m$. Suppose we wanted to compute its operator norm. One natural strategy would be to first find the singular value decomposition of $A$ and then extract the top singular value. However, this strategy is often too expensive for large matrices. Instead, here are two computationally friendly ways to approximate the operator norm of a matrix. Consider a random vector $x \sim \mathcal{N}(0, I_m)$.

**(a)** Show that, with probability one,

$$\lim_{k\to\infty} \|(A^\top A)^k x\|_2^{1/k} = \|A\|_{\mathrm{op}}^2.$$

**(b)** Consider an iterative method that starts at $x_0 = x$, and at each iteration updates

$$x_{k+1} \leftarrow \frac{A^\top A x_k}{\|A^\top A x_k\|_2}.$$

Show that with probability one, we have $\lim_{k\to\infty} \|A^\top A x_k\|_2 = \|A\|_{\mathrm{op}}^2$.

**(c)** Generate $A \in \mathbf{R}^{n\times m}$ a random matrix with i.i.d. entries where $A_{11} \sim N(0,1)$ with $n = 100$ and $m = 200$. Compute using $\sigma_1(A)$ an SVD decomposition and plot the errors of both methods as a function of $k$. That is, plot $\left|\|(A^\top A)^k x\|_2^{1/k} - \sigma_1(A)^2\right|$ and $\left|\|A^\top A x_k\|_2 - \sigma_1(A)^2\right|$ versus the number of iterations $k$.

## Problem 3 - Yet another variant of Davis-Kahan $\sin\theta$

Here is yet another variant that is often useful. Let $\|\cdot\|_{\mathrm{F}}$ be the Frobenius norm.

**Theorem 1.** *Let $M, M^\star \in \mathbb{R}^{n\times n}$ be symmetric with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ and ${\lambda^\star}_1 \geq \cdots \geq \lambda_p^\star$, respectively, and let $E = M - M^\star$. Fix integers $1 \leq r \leq s \leq n$ and assume*

$$\Delta := \min\{\lambda_{r-1}^\star - \lambda_r^\star, \lambda_s^\star - \lambda_{s+1}^\star\} > 0, \quad \textit{where by convention } \lambda_0^\star := +\infty, \textit{ and } \lambda_{n+1}^\star := -\infty.$$

*Let $d := s - r + 1$, and let*

$$U := [u_r, \ldots, u_s] \in \mathbb{R}^{n\times d}, \quad U^\star := [u_r^\star, \ldots, u_s^\star] \in \mathbb{R}^{n\times d}, \quad \Lambda = (\lambda_r, \ldots, \lambda_s), \quad \Lambda^\star = (\lambda_r^\star, \ldots, \lambda_s^\star),$$

*where $M u_j = \lambda_j u_j$ and $M^\star u_j^\star = \lambda_j^\star u_j^\star$ for $j = r, \ldots, s$. Further, let $U_\perp \in \mathbf{R}^{n\times(n-d)}$ and $U_\perp^\star \in \mathbf{R}^{n\times(n-d)}$ be such that $[U, U_\perp] \in O(n)$ and $[U^\star, U_\perp^\star] \in O(n)$. Then,*

$$\left\|U_\perp^\top U^\star\right\|_{\mathrm{F}} \leq \frac{2\|E\|_{\mathrm{F}}}{\Delta}.$$

The proof is somewhat similar to the one in class, and you will develop it in this exercise.

**(a)** Show that $\|U\Lambda^\star - M^\star U\|_{\mathrm{F}} \leq \|EU\|_{\mathrm{F}} + \|U(\Lambda - \Lambda^\star)\|_{\mathrm{F}}$. Use this inequality to conclude $\|U\Lambda^\star - M^\star U\|_{\mathrm{F}} \leq 2\|E\|_{\mathrm{F}}$, feel free to use the following fact without a proof.

**Fact 1** (Hoffman-Wielant Inequality)**.** *Let $A, B \in \mathbb{R}^{n\times n}$ be symmetric. Then,*

$$\|\lambda(A) - \lambda(B)\|_2 \leq \|M - M^\star\|_{\mathrm{F}},$$

*where $\lambda(\cdot)$ is the vector sorted eigenvalues of its input.*

*(b) Prove that $\|U\Lambda^\star - M^\star U\|_{\mathrm{F}} \geq \left\|{U_\perp^\star}^\top U\Lambda^\star - \Lambda_\perp^\star {U_\perp^\star}^\top U\right\|_{\mathrm{F}}$. **Hint:** Pythagoras always comes to the rescue.*

*(c) We will use a simple linear algebra fact that requires some additional notation. Let $A \in \mathbf{R}^{n\times m}$ and $B \in \mathbf{R}^{p\times q}$, then their Kronecker product is a matrix $(A \otimes B) \in \mathbf{R}^{pn\times qm}$ defined by blocks via*

$$A \otimes B = \begin{bmatrix} A_{11}B & \cdots & A_{1m}B \\ \vdots & \ddots & \vdots \\ A_{n1}B & \cdots & A_{nm}B \end{bmatrix}.$$

**Fact 2.** *The map that sends a matrix $W$ to the matrix product $W \mapsto BWA^\top$ can be written as $BWA^\top = (A \otimes B)\operatorname{vec}(W)$ where* vec *stacks the columns of $W$ into a vector.*

*Use this fact to show that* $\left\|U_\perp^{\star\top} U \Lambda^\star - \Lambda_\perp^\star U_\perp^{\star\top} U\right\|_{\mathrm{F}} \geq \Delta \|U_\perp^\top U^\star\|_{\mathrm{F}}$.

*(d) Leverage these inequalities to prove this version of the Davis-Kahan $\sin\theta$ theorem.*

## Problem 4 - Fixing the spectrum

In Lecture 13 we proved a guarantee for the recovery of communities in the stochastic block model. This guarantee requires the probability of a link across communities $q$ to be sufficiently large. In this exercise, we will fix this issue. Let $G \sim G(n, p, q)$ be a graph drawn using this distribution and $A$ be its random adjecency matrix.

**(a)** Consider the following quantity

$$\widehat{\lambda}_1 = \frac{2}{n(n-1)} \sum_{i<j} A_{ij}.$$

Show that with high probability (i.e., the probability goes to one as $n \to \infty$)

$$\left|\widehat{\lambda}_1 - \frac{p+q}{2}\right| \leq \frac{\log(n)}{n}.$$

**(b)** Design a modified spectral algorithm such that misclassifies at most $C/(p-q)^2$ misclassified vertices with high probability. **Hint:** Consider $\widehat{A} = A - \widehat{\lambda}_1 \mathbf{1}\mathbf{1}^\top$.